FINANCIAL NETWORKS AS DIRECTED CYCLIC GRAPHS - DRAFT

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Abstract

Financial networks' study and understanding has become extremely important since the global financial meltdown in 2007 – 2009 when the inter-connectedness of institutions has surfaced as one of the major culprits for the magnitude of the distress. This paper aims at providing a new approach to describe and better understand the networks of institutions and their global properties. It is based on Directed Cyclic Graphs - a subset of Probabilistic Graphical Models which have already found use in other domains such as physics and computer science. The paper draws some parallels and contrasts with other studies in the field of Network Theory. It then concludes with a stylized example.

Keywords: Financial Networks, Cyclic Graphs, Network Theory

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1. Introduction and Motivation

The study of financial networks is not new and neither the awareness of its importance. In late 2009 the European Central Bank (ECB) hosted a workshop called "Recent advances in modelling systemic risk using network analysis" (see ECB (2010)) which gathered practitioners and academics from around the world to share and discuss advances in network theory. At that time the organization of such discussion can be seen a little bit as *post factum* given that the contagion started by a few defaulted SIFIs had already spread. The importance of the discussion was not to acknowledge that the world is vastly interconnected - this is a well known fact - but rather to attract the attention around the need of more systematic investigative approach to the properties and sources of instabilities that such interconnectedness can entail[16].

This paper contributes to the existing literature by introducing the concept of cyclical graphs and their properties to the study of financial networks. Directed Cyclic Graphs (DCG) is a subset of the more general toolkit of Probabilistic Graphical Models which has already found applications in Finance, Engineering, Computer Science and Medicine. Unlike Bayesian Networks (BN) and Markov Random Fields (MRF), they allow both directed and undirected edges in the graph as well as *cycles*. This provides a natural representation of a network of debt relations where an institution can have debt with a chain of other institutions and some of them can be in turn indebted with it, something that cannot be represented by acyclic graphs[17]. Neither a structure that allows cycles but with fully undirected edges (like MRF) can be satisfactory since it precludes accounting for *interventions* and *manipulations* in the network.

The model we are going to introduce here is a simple static one period model which will provide us with the *distribution* of defaults, let's say, over 1 year horizon[18], given the mutual debt structure in the network. We thus ignore the complication of a dynamic multi-period model which can complicate the entire apparatus by introducing difficult to calibrate, difficult to manage parameters (e.g volatility of the assets, re-configuration of the debt in each period). We believe that important messages can be distilled in the simple setting here and that can be obscured by introducing additional parameters[19].

In Section 2 we review some of the existing literature on the topic and try to distinguish between some different strands of thought. In Section 3 we introduce cyclic graphs and give a simple example in Section 4. We treat the general case in Section 5 and describe interventions and give a more complex example in Sections 6 and 7. We then conclude.

2. Literature Review

The recent paper of Acemoglu et al (2013) studies the network properties of a group of interconnected banks and tries to answer the normative question of which network connectivity and under which conditions,

[16] policy makers are currently not only refining the regulatory and institutional set-up, but also looking for new analytical tools that help to better identify, monitor and address sources of systemic risk" in ECB (2010).

[17] The word institution can be interpreted in the broader sense of a company, not necessarily a financial institution

[18] This is the horizon for the banking book RWA in Basel II and the horizon of rating models. Other choices are, of course, acceptable

[19] We will, however, show a recipe of how to make an extension to a multi-period setting

provides the most resilient setup. The conclusions are in the middle of two opposing views in some previous papers like Allen (2000) and Vivier-Lirimont (2006) that claim that either a better bank interconnectivity can play a stabilizing or destabilizing role[20]. In this paper we will adopt a *descriptive* point of view but will show a method to study a financial network *as is*, by also making use of market implied information. We will show a *distribution* of the different defaults configurations.

Cont et al (2010) analyze a realistic connectivity case - that of the Brazilian banking system. They examine the distribution and the properties of connections, such as fat-tails and concentration. Once the networks structure is completely crystallized by the data, it is subjected to macroeconomic stress scenarios by shocking the capital buffer an institution i by an amount $\varepsilon_i(Z)$ where Z is the common macroeconomic shock. Thus a market shock affects the capital of all institutions in the network, which makes it more vulnerable to potential losses and increases the likelihood of large default cascades i.e domino effects.

A slightly different point of view is the one presented in Filiz et al (2012). Filiz et al. adopt Markov Random Fields (MRF) as a tool to study financial networks. MRFs have a straightforward application in physical systems where they can used to model a system of interacting atoms - the Ising model. Similarly, a financial network can be seen as a system of atoms (and the interactions between them) where the default of a set of debtors institutions to institution i, can 'flip' i into default. Relations between institutions are hardly symmetrical and there is always a net debtor/creditor. By postulating a MRF we assume a symmetry. One may argue that, as long as, we are interested in obtaining a joint probability distribution it does not matter. However, as stated in the Introduction, MRFs, apart from being a less intuitive non-directional representation, cannot account for manipulations in the network. As Schmidt et al. (2009) point out undirected graphs do not offer the possibility to distinguish between `seeing' and `doing'. Therefore, we cannot answer queries of the type: what would happen if a set of institutions defaults (or is forced to default) for reasons external to the debt structure. An institution may go bankrupt for a series of reasons which are exogenous. An institution may also be bailed-out by a government and 'forced' not to default. Unlike, Filiz et al. we will model a financial network as-is, and not hierarchically by inserting intermediate levels for sectors because such setup is not necessary for what we will want to show.

3. Directed Cyclic Graphs

Probabilistic graphical models (PGM) are a very useful way of decomposing complex distributions in smaller interacting parts. PGM can be either directed, undirected or both (hybrid). Directed graphs with no cycles or Directed Acyclic Graphs (DAG)[21] are also known as Bayesian Networks. Their introduction in Risk Management and Asset Allocation is due to Rebonato (2010) and later Rebonato and Denev (2012, 2013). DAGs are very useful in telling temporal stories. In fact, there is no way an event in the future in a given scenario can influence an event in the past[22]. However, acyclic graphs can be limited in their representational capacity in certain applications, especially when treating spatial relationships. For example, if we consider the network of debt between companies there is always the possibility of the 'flow' of debt relations to come again to a given company. For example, the companies in Fig.1 have a cyclical debt structure[23]. This means that X is indebted Y, Y is indebted Z and Z is indebted X. The nodes of the network are *boolean random variables* which represent that company defaulting or not[24].

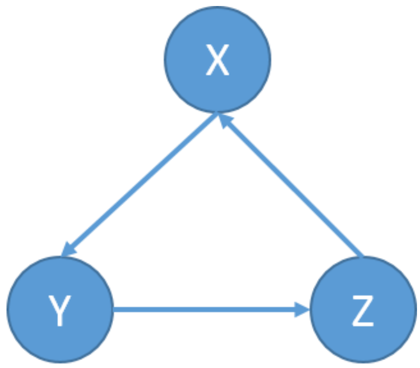


Figure1. A cyclic debt relationship

Of course, cyclical nets include acyclical nets as a particular case and, if there are not cyclical relations in a structure, we can use DAGs and their semantics. Cyclical graphs are well described in Schmidt (2009). The advantage of cyclical nets with respect to Markov Random Fields (MRF) is that they allow interventions and manipulations, as we will explain later.

Our aim is to define a joint probability distribution of a graph which will give the probability of different combinations of companies defaulting or not. So, for the example in Fig.1, we have 2^3 combinations and for each of them we have to deduct the probability of occurring. The task, in general, is

[20] See also the paper of Gai (2010)

[21] DAGs are constituted from a collection of vertices (nodes) and directed edges, each edge connecting one vertex to another, with no cycles which means that there is no way to start from a vertex V_i and follow a sequence of edges that leads back to V_i.

[22] NB This is a very different statement from the one that events of today incorporate expectations about the future

[23] The orientation of the arrow is that of the net debt. We are supposing here a netting agreement in place

[24] This is not a very strict requirement as nodes can be *multinomial* and express states such as e.g. distress, severe-distress, insolvency

made easier by virtue of the sparsity of connections that we can have in a more realistic structure compared to the fully connected one in Fig.**Ошибка! Источник ссылки не найден.**. We follow Schmidt (2009) in defining the joint distribution over a set of discrete variables X_i (for $i = 1, \dots, N$) in a DCG as a globally normalized product of non-negative interventional potential functions:

$$p(X_1, X_2, \dots, X_N) = \frac{1}{Z} \prod_{i=1}^{N} \varphi(X_i | X_{\pi(i)}) \quad (1)$$

where $\pi(i)$ is the set of parents of i[25]. This means that we have to assign quantities that depend on the parents only i.e. local interactions. Formula (1) will allow us to glue them to obtain a global quantity - the joint probability. The potential is a positive real valued function over a discrete domain Ω:

$$\varphi(\Omega): \Omega \to \mathbb{R}^+ \quad (2)$$

in this case the domain is the discrete set of possible configurations of each node given its parents. The normalization constant Z is given by:

$$Z = \sum_{X_1, X_2, \dots, X_N} \prod_{i=1}^{N} \varphi(X_i | X_{\pi(i)}) \quad (3)$$

Table 1. Balanced sheet of company X with total assets = 85 $

-	Assets	Liabilities
X	65	35
Y	20	50
Z	0	0

Table 2. Balanced sheet of company Y with total assets = 100 $

-	Assets	Liabilities
X	50	20
Y	50	5
Z	0	75

Table 3. Balanced sheet of company Z with total assets = 75 $

-	Assets	Liabilities
Y	75	0
Z	0	75

We define *Markov blanket* $MB(X_i)$ of a node X_i in a Cyclic Graph as the set of nodes composed of X_i's parents, children, and the parents of its children. $MB(X_i)$ separates X_i from all other nodes in the graph G which means that all the information we need for a node is the configuration of its blanket which screens it from what happens in the rest of the graph. This means that:

$$p(X_i | X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_N) = p(X_i | MB(X_i))$$

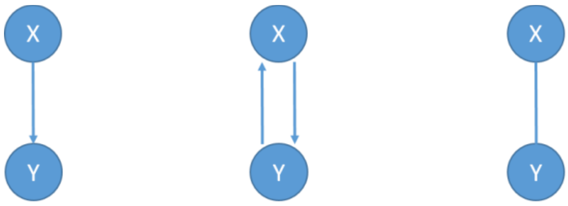
No subset of $MB(X_i)$ has this property. This allows achieving significant savings in terms of the parameters to provide.

4. Why Directed Cyclic Graphs

We can start illustrating our point with a simple network of 3 nodes only. Consider the nodes representing 3 institutions X, Y and Z which have equity and some mutual amount of debt. Their balance sheets can be represented as those shown in Tables 4[26], 4 and 4. The balance sheet of the third institution Z is very simple since we want to focus mainly on the relationship between X and Y[27].

Each row shows the asset/liabilities held by the company itself and the other company. So, for example, company X will have own assets such cash, bonds, immovable assets etc. plus receivables owed by company Y to company X. The liabilities, on the other hand, will also be composed of what owed by company X to company Y and an equity stake 'owned' by X itself. These balance sheets are consistent from an accounting point of view as can be easily verified.

Now, what if company X defaults? This will have a net effect on the balance sheet of Y of increasing its probability of default since the assets will go down by the amount owed by X minus the recovery rate RR. If the assets shrink, also the equity will do so by a certain amount[28]. Of course, in a low-concentration and well diversified portfolio we expect the effect of one obligor default to be negligible.



a) b) c)

[25] The pair of nodes connected by undirected edges are called *neighbors*. A set of nodes from which there is an arrow pointing to another one are called its *parents*. A set of nodes reached by an arrow from another one are called its *children*, which together with their neighbors and their children are called *descendants*. Since we will deal with DCGs (although we will continue mentioning MRFs) the notion of neighbor is not strictly needed

[26] The asset side of this balance sheet shows what is owed by other companies to X. The amount owed by X to X is the amount of assets of X which stay in the company. The liability side shows what X owes to the others. The amount owed to X by X is the equity

[27] An example involving only X and Y would have been excessively simplistic

[28] It will depend on how much also the liabilities will be reduced by netting

Figure 2. a) A simple two node acyclic BN b) A simple two node BN with a feedback loop (cyclic) c) A MRF

We can represent the relationship between X and Y as the graphical model shown in Fig. 2a[29]. Each node is boolean random variable representing the event the default of that entity. So $P(X)$ will be the probability of default (PD) of X[30]. In this representation X owes money to Y and not vice versa. This translates in the probability of default of Y being greater in the case of default of X i.e. $P(Y|X) > P(Y|\tilde{X})$, where by $\tilde{X}$ we mean the negation of the event 'default of X'. By virtue of the Bayes theorem we have that:

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)} = \frac{P(Y|X)P(X)}{P(Y|X)P(X)+P(Y|\tilde{X})P(\tilde{X})}$$

and hence in general:

$$P(X|Y) \neq P(X)$$

this means that the default of Y has also influence on X. But how is it possible that a default of Y can influence X, given that Y does not owe any net amount to X? We have to be careful of how we interpret directed edges. In fact, they are endowed with causal as well as with diagnostic probabilistic interpretation i.e. inference opposite to the direction of an arrow. Here the legitimate question is: given that we know that Y has defaulted, how much does this increase our belief that also X has defaulted? Well, if we know that Y defaulted this must certainly increase our faith in the fact that also X might have defaulted[31].

In the case of absence of netting agreement[32], we can model the interaction with a feedback loop as in Fig. 2b. This may seem as overkill since ultimately in a court settlement most probably, with some provisions, offsetting positions will be netted. This could, however, take time in which no cash flows exchange hands and be with an uncertain outcome. Sometimes, if we are not interested in manipulations of the network, we can collapse the opposite arrows in an undirected edge as shown in Fig. 2c.

A default of X does not necessarily means a default of Y: $P(Y|X) \neq 1$. In fact, some of the assets of X can be still recovered after default. When assigning a number we have to respect the sensible constraint that $P(Y|X) > P(Y|\tilde{X})$ but we can use also some market implied information e.g. a CDS price or a credit bond spread to deduct the marginal probability of X and Y defaulting[33]:

$$P(Y) = P(Y|X)P(X) + P(Y|\tilde{X})P(\tilde{X})$$

This leaves with 1 more parameter to estimate out of the 3 parameters needed to estimate a joint probability distribution[34]. For $P(Y|X)$ we can use a scoring model (or a Merton mode, for example) which assigns a PIT PD according to a leverage which will obviously increase in case the asset side of the balance sheet is downsized more than the liability side. This means that more a company contributes to the receivables of another, the more impact it will have in case of default[35]. The aim of this paper is not to pinpoint exact estimates of PDs - that is the role of rating models - but to show network effects. However, the guidelines we have just given narrow down the space of the available values for the parameters to estimate.

An important question to answer is why we believe that a probabilistic, and not deterministic model of network interactions, would yield a more appropriate description. In first place, in the case of default the recovered amount, given by a LGD distribution, is random quantity. We know that it defined between 0% and 100%[36]. This means the the asset side of a lender would shrink by a number between 0 and the exposure at default (EAD). Second, the asset shrinkage caused by an obligor's default will decrease the equity (when written-off) and so increase the leverage, but the leverage is just one of the determinants of a company default. In a rating model it is one of the most important factors but not the only one. Liquidity, profitability, debt-servicing ratios also play their role. In the Merton model, for example, it is also the volatility of the assets that matters, not just the leverage. Let's remind that the aim of this paper is to study the effect of the network of debt on the mutual *local* increase/decrease of probabilities of default and be able to deduct the $global$ properties of the network, regardless of the way to estimate the local PDs, so will not dig deep in this aspect although we have (and we will) provide useful pointers. Let's start by introducing some formalism in the next section.

5. The General Case

We will show how to deal with more complicated structures as the one shown in Fig. 3. Given a network of N companies, we will denote by A and L the asset and liabilities matrices that determine the debt structure. Each row i of A will denote the percentage

[29] We don't need the entity Z for the point we will try to make

[30] Being this a boolean variable means that the probability of non-default equals $P(\tilde{X}) = 1 - P(X)$.

[31] Of course, company Y can default for other reasons. For example, invest its assets which are not receivables (50) in a project which could fail and thus lose the entire amount

[32] Such as e.g. ISDA Master Agreement and/or netting of loans and deposits and/or AR/AP netting

[33] Of course, the CDS price does not give directly a PD as it is in the risk neutral measure

[34] 2 boolean nodes need $2^2 - 1$ parameters. The -1 comes from the normalization condition for probabilities

[35] Another option is to use the equity correlations to proxy for the asset correlation and hence the default correlation. See Eq. 11 for how default correlation is linked to $P(X,Y)$, $P(X)$ and $P(Y)$. Attention must be paid to use this market based approach as the parameters estimated in this way contain information both about the market *expectations* and *risk aversion*

[36] Empirical evidence shows that the distribution has high probability masses around 0% and 100%.

of debt owed to i by each of of the companies in the columns j. The element (i,i) of A will represent the assets of i held by i i.e. that part of the assets is not lent to other companies e.g. cash, buildings etc. Similarly, each row i of L will show what percentage of the liabilities is owed to each company in the columns. The element (i,i) will represent the equity of i[37]. The following identities must hold[38]:

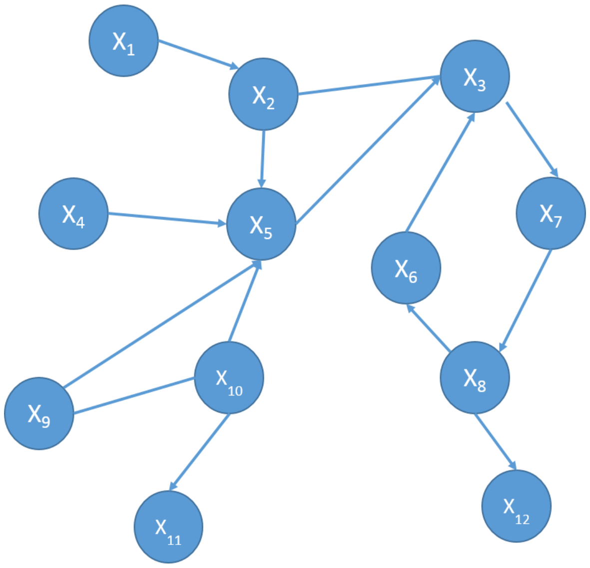


Figure 3. A more complex network of debt dependencies

$$\sum_j A_{ij} = 1 \qquad \sum_j L_{ij} = 1 \quad \forall i \qquad (8)$$

which are the normalization conditions for each row and by rewritting in terms of debt and equity:

$$E_i + \sum_{j \neq i} L_{ij} = 1 \quad \forall i \qquad (9)$$

we will introduce the two vectors As and Li whose components will be the asset and liabilities of each company i in dollar amount. The following is an accounting identity:

$$As_i = Li_i \quad \forall i$$

and obviously:

$$As_i \sum_j A_{ij} = Li_i \sum_j L_{ij} \qquad (10)$$

we also know that the liabilities of j must be 'come' from somewhere i.e. from the assets of all other companies. This is expressed through:

$$\sum_{i \neq j} As_i A_{ij} = Li_j - E_j = D_j \qquad (11)$$

and:

$$\sum_{i \neq j} Li_i L_{ij} = As_j - As_j A_{jj} \qquad (12)$$

If company i defaults we have to revise the i-th column of matrix A in those rows that correspond to the non-0 columns of L and the i-th column of the L matrix. We have to revise also the row i of L. Symmetrically, we have to revise also the i-th row of A and all the non-0 rows of i-th column of L. Also the diagonal elements of L need to be updated. The leverage of a company j is given by:

$$Lev_j = \frac{As_j}{Li_j L_{jj}} = \frac{As_j}{Li_j(1 - \sum_k L_{jk})}$$

if the company i defaults we have to revise this number for each j as follows (assuming 0 recovery):

$$Lev_{j'} = \frac{As_j \sum_{k \neq i} A_{jk}}{As_j \sum_{k \neq i} A_{jk} - Li_j \sum_{k \neq i \neq j} L_{jk}}$$

It must be:

$$Lev_{j'} > Lev_j$$

if we calculate it in the direction of the arrows i.e. if i is a parent of j. This increase in leverage will be used to calculate conditional probabilities as will be discussed below.

Our next task is to specify a joint probability distribution given the local assignments. We have to make use of Eq. (1) to do so. One simplification comes immediately in mind. If a node is not part of a cycle and has only parents (i.e. no neighbors) and children which are not part of a cycle, we can use directly the language of conditional probabilities instead of potentials. For example, we can assign to the nodes X_{11} and X_{12} the quantities $P(X_{11}|X_{10})$, $P(X_{11}|\tilde{X}_{10})$, $P(X_{12}|X_8)$ and $P(X_{12}|\tilde{X}_8)$ without normalizing. Our life would be much easier had all the relations been of this type. In all other cases i.e. we have to assign a potential to account for unavoidable cycles and undirected edges. This might be cognitively difficult as the entire network has to be normalized through Z in Eq. (3) which couples the potentials assigned to the components of the graph. We can well think to equalize the potential to conditional probabilities but, as remarked above, this rarely will correspond to the 'final' probabilities, if not in a handful of cases, by virtue of the definition of Z.

It can be cognitively easier to assign probabilities rather unnormalized potentials. If this is the case, we can constrain the parameters of the potentials of the graph to a set of probabilities we are able to provide. To start with we can assign PDs to all

[37] In this formalism we include the equity in the liabilities side. When we will want to refer only to the debt structure we will use the symbol D and the equity structure by the vector E

[38] One entity which have to take into consideration in a more realistic scenario is the household sector who acts both as borrower and lender for the banking sector. We cannot model all the agents individually but we could, for example, assign one node X_0 to the entire retail sector. We will not treat the subtleties of this case here

the nodes taken from CDSs, external[39] or internal PIT rating systems. We can assign conditional probabilities $P(X_j|X_i)$ by calculating the effect on the leverage of j of the default i. It is always more intuitive to proceed in the direction of the arrow. In case of the lack of netting, one of the two directions can be chosen[40].

It is convenient to cast the potential in the following form[41]:

(9)
$$\varphi(X_i|X_{\pi(i)}) = exp\left(w_{X_i,X_i}X_i + \sum_{j\in\pi(i)} w_{X_i,X_j}X_iX_j\right)$$

where w_{X_i} is a scalar bias for each node i and w_{X_i,X_j} is the interaction weight between i and j. We remind that a capital letter X represents a random variable, and not its instantiation which we denote by small letter x. Therefore, w_{X_i} and w_{X_i,X_j} assume 1 and 2 values respectively[42] and they are 'activated' when the nodes they multiply are$= 1$. With the normalization constant Z given by:

$$Z = \sum_{i\in V, x_i\in X_i} exp\left(w_{X_i,X_i}X_i + \sum_{j\in\pi(i)} w_{X_i,X_j}X_iX_j\right)$$

We can provide the following information to specify the parameters w_{X_i,X_i} and w_{X_i,X_j}:

• For each i the marginal probability $P(X_i)$

• For each pair $\{i,j\}$ the conditional probability $P(X_i|X_j)$[43]

Is this information sufficient to pin point all the parameters? The first sanity check would be count the numbers of parameters we provide versus those we need to specify the DCG. They are, in fact, the same - for each w_{X_i,X_i} we associate $P(X_i)$ and for each w_{X_i,X_j} we have $P(X_i|X_j)$[44]. We note that trivially the set of statistics above is equivalent to the following set of constraints S:

• For each i the marginal probability $P(X_i)$

• For each pair $\{i,j\}$ the joint probability $P(X_i,X_j)$

In fact, $P(X_i,X_j) = P(X_i|X_j)P(X_j)$. Filiz et al. (2012) show (see Theorem 2.1) that there is a unique set of parameters w_{X_i,X_i} and w_{X_i,X_j} that match the set of constraints S for potentials of the form (9). We show a numerical example in Section 7[45].

6. Interventions

We have claimed that the advantage of DCGs over MRFs is in the possibility to intervene on the network. Let's first distinguish between *manipulation* and *observation*. If we happen to know that X_i has defaulted we can immediately update the joint probability by setting in it $X_i = 1$ and re-calculate $P(X_j|x_i = 1)$ for all the other nodes j and obtain a new joint probability. In our context, we do not observe defaults when we build a network. We usually build it on non-defaulted entities. Defaults will happen (or not) during the 1-year observation horizon. We have full information on all the entities at all times i.e. we do not have partial observations[46]. We can ask, however, the normative question of what would happen if we manipulate the network by forcing some Xs to take a certain value. If we are a central bank, we may want to know what would happen if we guarantee a bank or decide to remove its license and discontinue its activities e.g. if it has violated laws against money laundering and the financing of terrorism. Or simply an exogenous operational event can be uncovered e.g. Barings.

Table 4. Marginal probabilities for the nodes

$P(X)$	0.05
$P(Y)$	0.04
$P(Z)$	0.06
$P(T)$	0.05

Table 5. Conditional probabilities for the nodes

$P(Y|X)$	0.2
$P(Z|Y)$	0.1
$P(X|Z)$	0.15
$P(T|X)$	0.12

The rule is very simple. We remove $\varphi(X_i|X_{\pi(i)})$ from the Eq. (1) for the entity i and recalculate the rest of the product by setting X_i to the manipulated value. In the graph what we have to do is delete the

[39] Most of the institutions will have an external rating assigned by at least one of the 3 agencies

[40] NB For example, for $X - Y$, both $P(Y|X)$ and $P(X|Y)$ can be assigned but then we need just one more parameter - either $P(X)$ or $P(Y)$

[41] This is similar to *causal independence*. See Heckerman (1998).

[42] In general, w_{X_i,X_j} is not symmetric i.e. $w_{X_i,X_j} \neq w_{X_j,X_i} = 0$ in case of j being the parent and i the child. We can enforce symmetry in the case of a feedback loop to avoid redundancy of the parameters as we need only 3 of them to fully define the interaction

[43] $P(X_i|X_j)$ is linked to $P(X_j|X_i)$ by the Bayes theorem. If we know $P(X_i|X_j)$, $P(X_i)$ and $P(X_j)$ we can derive $P(X_j|X_i)$.

[44] We remind such association is only for enumeration purposes and, if this assignment was made, the $P(X_i)$ and $P(X_i|X_j)$ will lose the meaning of the probabilities we want them to represent because of the presence of Z

[45] We note in closing that the setup here is extendable to a multi-period case. We can assume that the network is re-configured at the end of each period by removing the defaulted nodes and recalculating everything on the remaining ones for the number of periods in the future we want. We could use the CDS curves of the institutions to deduct the PDs for the future periods (we remind again here the caveat of switching between Q and P measures). However, the mutual debt structure, which is the other determinant of the potentials, is unlikely to remain static in the future

[46] Like it can happen, for example, in medical diagnosis

arrows incoming to i but keep the outgoing ones. In case of feedback loops (or undirected edges that can be viewed as such) we delete only the incoming arrows to i.

7. An example

For presentation purposes we will limit ourselves to a stylized network of 4 banks. The reasoning behind is, however, universally applicable. We will show results for the graph in Fig. (4)a which contains one cycle $X \to Y \to Z \to X$ and one undirected edge $X - T$. We will show what changes after the intervention on node X shown in Fig. (4)b.

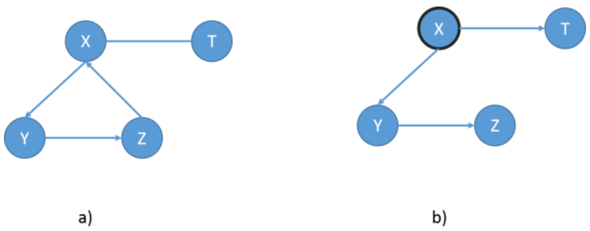


Figure 4. The graph used in the example a) before intervention on node X b) after intervention on node X

The probabilities we used are in Tables 7 and 7.

We found the parameters in Eq 9 which satisfy these probability constraints. We report them in Table 7. The w matrix in the Table has on the main diagonal the scalar bias for each node, while the couplings for the other nodes are the other elements of the matrix w_{X_i,X_j}. The calculation of the joint probability table (JPT) is straightforward and it gives:

Table 6. The matrix $w_{(xi,xj)}$

−3.35	0	1.32	0.52
2.00	−3.44	0	0
0	0.21	−2.88	0
0.52	0	0	−3.02

$$\begin{bmatrix} X & Y & Z & T & \text{JointProbability} \\ 0 & 0 & 0 & 0 & 0.8306 \\ 0 & 0 & 0 & 1 & 0.0403 \\ 0 & 0 & 1 & 0 & 0.0468 \\ 0 & 1 & 0 & 0 & 0.0268 \\ 1 & 0 & 0 & 0 & 0.0291 \\ 0 & 0 & 1 & 1 & 0.0023 \\ 0 & 1 & 0 & 1 & 0.0013 \\ 0 & 1 & 1 & 0 & 0.0019 \\ 1 & 0 & 0 & 1 & 0.0040 \\ 1 & 0 & 1 & 0 & 0.0061 \\ 1 & 1 & 0 & 0 & 0.0070 \\ 0 & 1 & 1 & 1 & 0.0001 \\ 1 & 0 & 1 & 1 & 0.0008 \\ 1 & 1 & 0 & 1 & 0.0010 \\ 1 & 1 & 1 & 0 & 0.0018 \\ 1 & 1 & 1 & 1 & 0.0002 \end{bmatrix} \quad (10)$$

The most probable event is 'nothing happens' in row 1[47]. We can count the defaults and show a bar chart statistics for the number of defaults as shown in Fig.5.

The distribution is just for 4 nodes and it is difficult to see fat tail effects. We can make some sanity checks, however. For example, if the events were independent we can calculate the probability of one and only one institution defaulting which is:

$$P(X)(1-P(Y))(1-P(Z))(1-P(T)) + (1-P(X))P(Y)(1-P(Z))(1-P(T)) + \ldots$$
$$\ldots + (1-P(X))(1-P(Y))P(Z)(1-P(T)) + (1-P(X))(1-P(Y))(1-P(Z))P(T)$$

and which gives 17.16% instead of 14.30% as calculated from the JPT. Similarly, for 0 defaults the results are 81.44% and 83.06%. This means that the probability of having more than 1 default in the 2 cases is 1.39% and 2.64% respectively. As expected, the independence case is more shifted to the left.

[47] Row 12 and 16 are not zero but the significant digits are after the fourth one

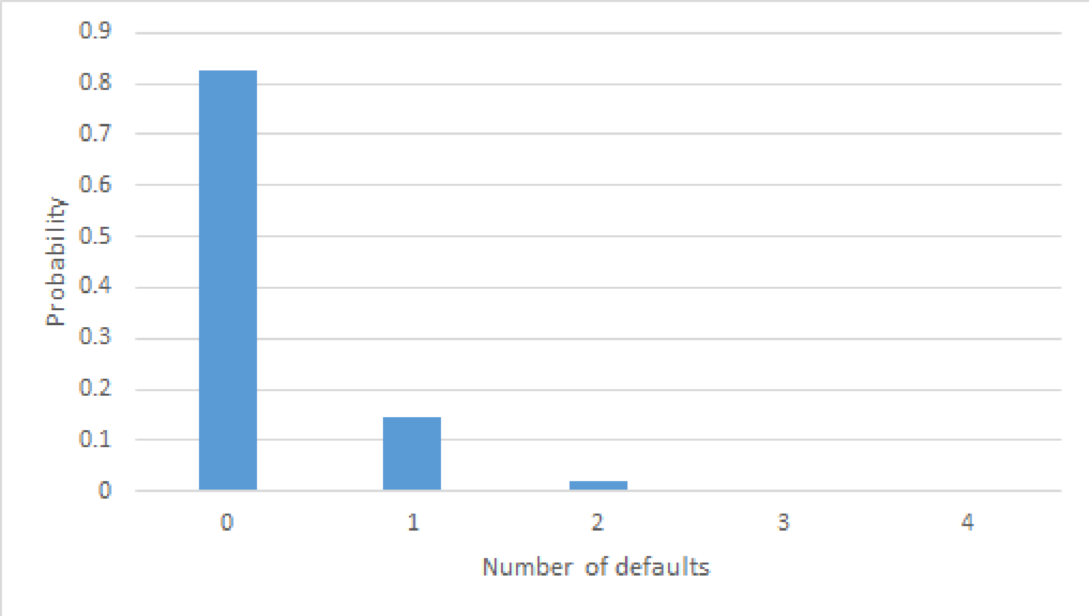


Figure 5. Distribution of the number of defaults

Table 7. Exposures of the four banks in $

E_X	5000
E_Y	1500
E_Z	2000
E_T	3000

Another useful statistics is the default correlation matrix for 2 variables X and Y as defined by the formula:

$$\rho_{ij} = \frac{P(X_i,X_j) - P(X_i)P(X_j)}{\sqrt{[P(X_i)(1-P(X_i)P(X_j)(1-P(X_j))]}} \quad (11)$$

and in this concrete case given by:

$$\begin{bmatrix} \text{DefaultCorrelation} & X & Y & Z & T \\ X & 1 & 0.19 & 0.12 & 0.07 \\ Y & 0.19 & 1 & 0.03 & 0.01 \\ Z & 0.12 & 0.03 & 1 & 0.01 \\ T & 0.07 & 0.01 & 0.01 & 1 \end{bmatrix}$$

Given the low values of the probabilities in the network also the default correlations are low. The node that is on average the more correlated with the others is node X. We expect it to play an important role in the distribution of losses.

In Table 7 we show the vector of exposures for the 4 banks. By assuming $RR = 0$ the loss distribution can be seen in Fig.6. The Expected Loss is $EL = 580\$$ while the loss at 95% confidence is $L_{95\%} \approx 4500\$$.

We then intervene on node X by setting it to 0 and trim the graph according to the rules in Section 6. The result is shown in Fig. (4)b. The Expected Loss is $EL = 314\$$ while the loss at 95% confidence is $L_{95\%} \approx 2000\$$. The loss at 95% confidence level is reduced by $\approx 2500\$$. Whether to intervene or not on the institution X would depend on whether the equity buffer to inject/guarantee is less then the loss on the entire network. The loss distribution is shown in Fig.7.

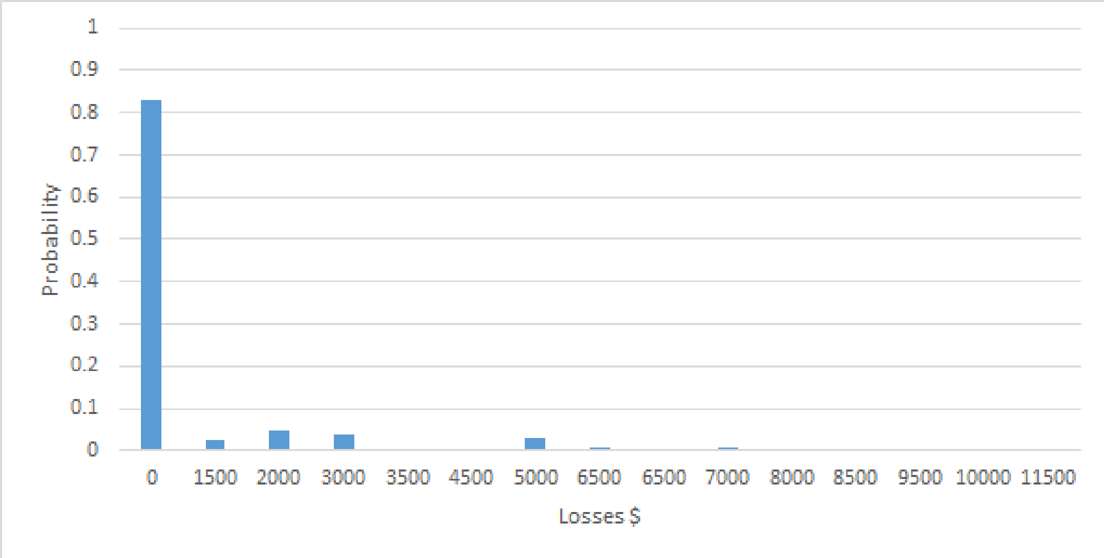


Figure 6. Distribution of losses

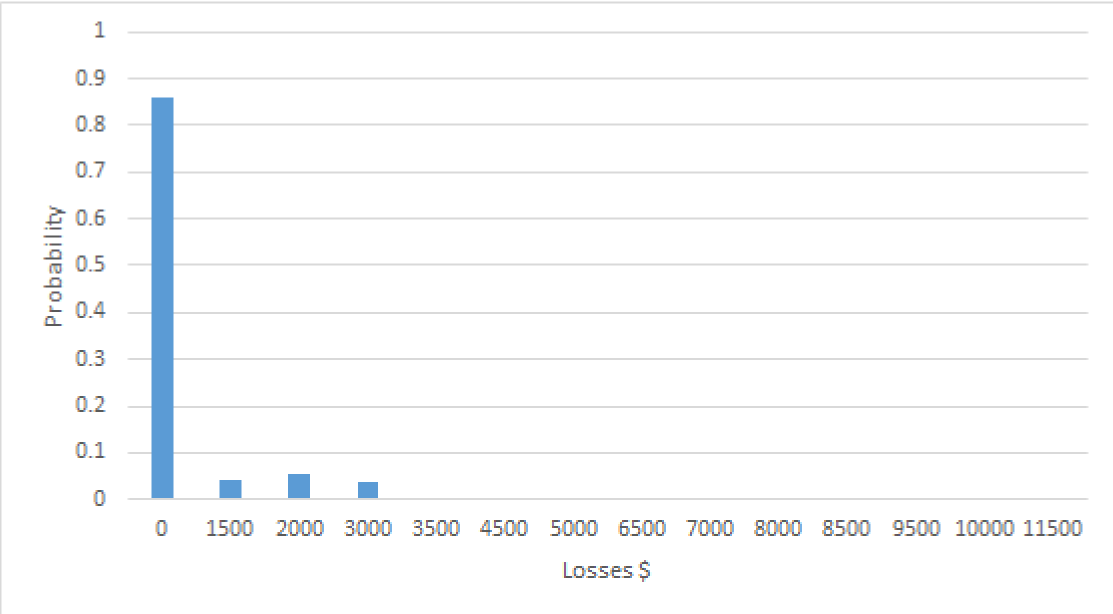


Figure7. Distribution of losses after the intervention on *X*

The example presented here is overly simplistic and a real world study could require the consideration of hundreds of companies. The inference problem becomes more burdensome but numerical methods such as Gibbs sampling exist to perform efficiently the task. See Koller (2009) for a review of different inference methods. JPTs can become of millions of lines but their row-by-row scrutiny becomes also unnecessary as summary graphs and distributions as those in Figs. 5 and 6 can be used to inspect the information.

We note in closing that we can input stressed parameters in the network and make inference also in that case. PDs, LGDs, EADs (or EPE) can be linked to macroeconomic variables such as credit cycles, FX and interest rates etc. We can assume certain values of those macrovariables under a stress scenario to obtain stress inputs to the network. We can then calculate a stressed joint probability table.

Conclusions

We introduced the theory of Directed Cyclical Graphs to the study of Financial Networks. We believe that such tool provides a good model of such networks as it is takes into account the directionality of influence and the existence of cycles in real world cases. The framework presented here allows for normative queries about how manipulating exogenously the network will propagate through it. Moreover, the contagion effects of one or more entities in default are easily inspectable in terms of joint probability tables.

The results are transparent (no black boxes!) and can be easily examined through visual tools and graphs of probability distributions.

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