EXECUTIVE STOCK OPTIONS WITH A REBATE: VALUATION FORMULA

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Abstract

We examine the valuation of executive stock option award where there is a rebate at exercise. The rebate depends on the performance of the stock of the corporation over time the period concerned; in particular we consider the situation where the executive can purchase the stock at exercise time at a discount proportional to the minimum value of the stock price over the exercise period. Valuation formulae are provided both when assessment is done in discrete time as well as in continuous time. Some numerical illustrations are also presented.

Keywords: executive stock option, rebate, geometric Brownian motion, Esscher transform, valuation formulae, numerical illustrations.

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1. Introduction

Stock options have become dominant component in executive compensation schemes in US and other industrial countries because they reward value creation better than other schemes such as bonuses tied to accounting results. Moreover they align the interest of executives with shareholders and attempt to retain talented executives who are in great demand. Generally executive stock options are pure vanilla call options with a longer term (about 5-10 years) and restrictions such as vesting. Valuation of executive stock options was not required until 2005 when FASB (Federal Accounting Standard Board) made it mandatory and provided guidelines, FASB123 (R), for expensing of executive stock options in the financial statements of corporations.

Nowadays there are many types of executive stock options such as indexed executive stock option, reload executive stock option. Some of the aspects of such executive stock options as related to corporate performance and governance can be found in Aggarwal and Samwick (1999b) and John and John (1993). As they are different from traded stock options, well known analytical formula of Black and Scholes does not provide appropriate valuation. Recent research to establish formulae to value executive stock options can be found in Hemmer et.al (1994), Kulatilaka and Marcus (1994) and Dayananda (2000). A coverage of papers related to executive compensation can be found in the text by Carpenter and Yermack (1998).

The award of executive stock options is now so widely made that even junior executives of corporations receive such offers. This paper is concerned about one form of widespread executive awards where the executive is allowed to purchase the stock at a specified time(exercise time) with a rebate, rebate depending on the performance of the stock; specifically rebate is a percentage (denoted by $\beta, 0 \le \beta \le 1$) and the total rebate is a product of the rebate and the lowest value of the stock during the specified period. Thus the payoff is similar to normal executive stock option. We present valuation formulae for such executive stock options with a rebate in a general framework in this paper.

2. Rebate assessed in continuous time

We assume that the stock price of the corporation at time t (≥ 0),S(t),follows a geometric Brownian motion so that

$$S(t) = S(0)\exp\{X(t)\} \qquad (2.1)$$

and that {X(t)} has a Brownian motion with mean and variance μ and σ^2 per unit time. Let the dividend rate of the stock be δ per unit time and the risk-free rate be r.

We assume that under the award, the executive will be allowed to purchase a stock at time t=τ at the price

$$S(\tau) - \beta Min\{S(s); 0 \le s \le \tau\} \qquad (2.2)$$

where $\beta (0 \le \beta \le 1)$ is called the rebate under the award.

Thus at time t=0, the value of one executive option would be

$$W = e^{-r\tau} E_Q[S(\tau) - \beta Min\{S(s); 0 \le s \le \tau\}] \qquad (2.3)$$

where expectation is under risk-neutral measure.

Let $U(\tau) = Min\{X(s); 0 \le s \le \tau\}$. (2.4)

Then (2.3) simplifies to

$$W = S(0) - \beta S(0) e^{-r\tau} E_Q[e^{U(\tau)}] \quad (2.5)$$

Using the Discussion by Gerber & Shiu in Tiong (2001) the density of the random

variable M(τ) where $M(\tau) = Max\{X(s); 0 \le s \le \tau\}$ is given by

$$f_M(y) = \frac{1}{\sigma\sqrt{\tau}}\phi(\frac{y-\mu\tau}{\sigma\sqrt{\tau}}) + e^{2\mu y/\sigma^2}\frac{1}{\sigma\sqrt{\tau}}\phi(\frac{-y-\mu\tau}{\sigma\sqrt{\tau}}) - \frac{2\mu}{\sigma^2}e^{2\mu y/\sigma^2}\Phi(\frac{-y-\mu\tau}{\sigma\sqrt{\tau}}); y \ge 0. \quad (2.6)$$

Since $U(\tau) = -Max\{-X(s); 0 \le s \le \tau\}$, its density is derived as

$$f_U(u) = \frac{1}{\sigma\sqrt{\tau}}\phi(\frac{-u+\mu\tau}{\sigma\sqrt{\tau}}) + e^{2\mu u/\sigma^2}\frac{1}{\sigma\sqrt{\tau}}\phi(\frac{u+\mu\tau}{\sigma\sqrt{\tau}}) + \frac{2\mu}{\sigma^2}e^{2\mu u/\sigma^2}\Phi(\frac{u+\mu\tau}{\sigma\sqrt{\tau}}); u \le 0. \quad (2.7)$$

We state the following lemma without proof.

Lemma

Let X be a random variable with mean μ_1 and variance σ_1^2. Then for any real θ and a

$$E[e^{\theta X} I(X > a)] = \Phi(\frac{\mu_1 + \sigma_1^2\theta - a}{\sigma_1})\exp\{\mu_1\theta + \sigma_1^2\theta^2/2\}. \quad (2.8)$$

Now the value of the award in (2.5) can be represented as

$$W = S(0) - \beta S(0) e^{-r\tau} E[e^U I(U \le 0; h^*)] \quad (2.9)$$

where h is the parameter under Esscher transform and

$$\mu + h^*\sigma^2 = r - \delta - \sigma^2/2. \quad \text{(see Appendix)} \quad (2.10)$$

Since the density for $U(\tau)$ in (2.7) has three terms the expectation in (2.9) would have three terms and so we define the following.

Let $A_i; i = 1,2$ and 3 be the ith term in the expectation of the expression in (2.9) using the density function in (2.7). Then using the result of the lemma directly

$$A_1 = \exp(-r\tau)\int_{-\infty}^{0} e^u \phi(\frac{-u+\mu^*\tau}{\sigma\sqrt{\tau}})\frac{1}{\sigma\sqrt{\tau}}du = e^{-r\tau}E[e^{-Y}I(Y_1 \ge 0)]$$

where Y_1 is a normal random variable with mean $-(r-\delta-\sigma^2/2)\tau$ and variance $\sigma^2\tau$. Thus ,we have

$$A_1 = \exp(-\delta\tau)\Phi\{((u+\sigma)\sqrt{\tau}\} \quad (2.11)$$

where $u = \frac{r-\delta}{\sigma} - \frac{\sigma}{2}$,

Similarly

$$A_2 = e^{-r\tau}\int_0^\infty e^{-ky}\frac{1}{\sigma\sqrt{\tau}}\phi(\frac{-y+r-\delta-\sigma^2/2}{\sigma\sqrt{\tau}})dy = e^{-r\tau}E[e^{-kY_2}I(Y_2 \ge 0)]. \quad (2.12)$$

where $k = 2\frac{(r-\delta)}{\sigma^2}$ and Y_2 is a random variable with mean $(r-\delta-\sigma^2/2)\tau$ and variance $\sigma^2\tau$. Using the lemma again, we have

$$A_2 = \exp[\{k\delta + k(k+1)\sigma^2/2 - (k+1)r\}\tau]\Phi\{(u-k\sigma)\sqrt{\tau}\}. \quad (2.13)$$

To evaluate the third term we use the fact that under Esscher measure(risk neutral) the corresponding stochastic process for {X(t)} with mean μ and variance σ^2 per unit time is transferred to another Weiner process with mean $\mu^*\tau$ and variance $\sigma^2\tau$ where $\mu^* = \mu + h^*\sigma^2 = r - \delta - \sigma^2/2$. Thus the third term of the expectation is given by

$$A_3 = \exp(-r\tau)2\frac{(r-\delta-\sigma^2/2)}{\sigma^2}\int_{-\infty}^{0}\exp\{(\frac{2\mu^*}{\sigma^2}+1)u\}\Phi(\frac{u+\mu^*\tau}{\sigma\sqrt{\tau}})du. \quad (2.14)$$

where $\mu^* = r - \delta - \sigma^2/2.$

Using integration by parts expression in (2.14) is reduced to

$$A_3 = 2e^{-r\tau}(\frac{r-\delta-\sigma^2/2}{\sigma^2})[\frac{1}{k}\Phi(\frac{\mu^*}{\sigma}\sqrt{\tau}) - \frac{1}{k}E\{e^{-kY_2}I(Y_2 \ge 0)\}]$$

which simplifies to

$$A_3 = \frac{2}{k}(\frac{r-\delta-\sigma^2/2}{\sigma^2})[e^{-r\tau}\Phi(u\sqrt{\tau}) - A_2]. \quad (2.15)$$

Hence the value at t=0 an award of one rebate option is given by

$$W = S(0) - \beta S(0)[e^{-\delta\tau}\Phi((u+\sigma)\sqrt{\tau}) + \frac{2u}{k\sigma}e^{-r\tau}\Phi(u\sqrt{\tau}) + (1-\frac{2u}{k\sigma})\exp\{(k(k+1+\delta)\frac{\sigma^2}{2} - (k+1)r)\tau\}\Phi((u-k\sigma)\sqrt{\tau})]. \quad (2.16)$$

when assessment is done in continuous time.

3 Rebate assessed in discrete time

The component of the rebate may depend on the stock price at specified points of time, for example on 1st January each year so that actual total rebate is assessed by examining the minimum value of the underlying stock price on 1st January of each year during the period concerned.

We assume that the rebate is assessed based on the price of the stock at discrete times $\tau_1, \tau_2, ..., \tau_m = \tau (0 \le \tau_1 \le \tau_2 \le .... \le \tau_m = \tau)$. The price to be paid at t= τ is given as

$$S(\tau) - \beta Min\{S(\tau_i); i = 1,2,..,m\} \quad (3.1)$$

Special case m=2

Then the value at time t=0 is given by

$$W = e^{-\tau} E_Q[S(\tau) - \beta Min\{S(\tau_1), S(\tau)\}] \quad (3.2)$$

where the expectation is with respect to risk-neutral measure.

Then using Esscher transform in the Appendix, its value can be represented as

$$W = S(0) - \frac{\beta S(0)e^{-r\tau}}{E[e^{h^* X(\tau)}]} E[e^{h^* X(\tau)} S(\tau_1) I(S(\tau_1) < S(\tau)) + e^{h^* X(\tau)} S(\tau) I(S(\tau) < S(\tau_1))],$$
$$= S(0) - \beta \frac{S(0)e^{-r\tau}}{E[e^{h^* X(\tau)}]} E[e^{h^* X(\tau) + X(\tau_1)} I(X(\tau) > X(\tau_1)) + e^{(h^* + 1)X(\tau)} I(X(\tau) < X(\tau_1))]. \quad (3.3)$$

Let $X = X(\tau) - X(\tau_1)$ and $X_1 = X(\tau)$.

Then X and X_1 are independent normal random variables.

Hence

$$W = S(0) - \beta \frac{S(0)e^{-r\tau}}{E[e^{h^* X(\tau)}]} E[e^{h^* X + (h^* + 1)X_1} I(X > 0) + e^{(h^* + 1)(X + X_1)} I(X \le 0)],$$
$$= S(0) - \beta \frac{S(0)e^{-r\tau}}{E[e^{h^* X(\tau)}]} [E\{e^{h^* X + (h^* + 1)X_1}\} P\{X > 0; h^*\} + E\{e^{(h^* + 1)(X + X_1)}\} P\{X \le 0; h^* + 1\}]. \quad (3.4)$$

Simplifying (3.4), we get the value of the rebate option when m=2

$$W = S(0)[1 - \beta\{e^{-r\tau_1} \Phi(u\sqrt{(\tau - \tau_1)}) + \Phi(-(u + \sigma)\sqrt{(\tau - \tau_1)})\}]. \quad (3.5)$$

General case m>2.

Using (3.1) and Esscher transform in Appendix, the option value at t=0 becomes

$$W = S(0) - \beta e^{-r\tau_m} \sum_{i=1}^{m-1} \frac{E[e^{hX(\tau_m)} S(\tau_i) I(S(\tau_i) < S(\tau_j); j \ne i, j = 1,2,....,m].}{E[e^{hX(\tau_m)}]} \quad (3.6)$$

We now define

$$w_i = X(\tau_i) - X(\tau_{i+1}) \text{ and } Y_{p,q} = \sum_{k=p}^{q-1} w_k = X(\tau_i) - X(\tau_q). \quad (3.7)$$

Then

$$W = S(0)[1 - \beta e^{-r\tau_m} \sum_{i=1}^{m-1} A_i] \quad (3.8)$$

where

$$A_i = \frac{E[e^{(h+1)V_i - hY_{im}} I(Y_{i,m} \le 0, Y_{i,m-1} \le 0,...Y_{i,j} \le 0,..., Y_{i,1} \le 0, i \ne j)}{E[e^{rX(\tau_m)}]}, \quad (3.9)$$

$V_i = X(\tau_i)$.

As the process{X(t)} has a Brownian motion and if $j_1 < j_2$ then

$$Cov\{Y_{i,j_1}, Y_{i,j_2}\} = \sigma^2(\tau_i - \tau_{j_1}) \quad (3.10)$$

else $Cov\{Y_{i,j_1}, Y_{i,j_2}\} = \sigma^2(\tau_i - \tau_{j_2})$.

Let Σ_i be the covariance matrix of the vector $\{Y_{i,m}, Y_{i,m-1},.., Y_{i,j}.., Y_{i,1}\}'; j \ne i.$ whose elements are given by (3.10).

Then the simplified form of option value at time t=0 is given by (3.8) where

$$A_i = e^{-r(\tau_m - \tau_i)} F[-u\sqrt{(\tau_m - \tau_i)}, -u\sqrt{(\tau_{m-1} - \tau_i)},..,-u\sqrt{(\tau_i - \tau_1)}; \Sigma_i],$$

and $F[x_1, x_2,...., x_{m-1}; \Sigma_i]$ is the distribution function of the multi-variate normal random vector $\{Y_{i,m}, Y_{i,m-1},.., Y_{i,j}.., Y_{i,1}\}'; j \ne i$ with covariance matrix Σ_i.

4. Some Numerical Illustration when assessment is in continuous time

We consider a typical case where S(0)=50, r=0.05, $\sigma = 0.25$, $\delta = 0.02, \tau = 10$. We find that the Black-Scholes value when the exercise price is the same as the stock price at grant is 17.38. We use the formula (2.16) and evaluate the stock option value W for different rebate values, β. The table below provides the values of W for $\beta = 0.05$ to 0.80 in steps of 0.05. It is observed that the option with rebate $\beta = 0.60$ is approximately equal to the Black-Scholes option value where the exercise price is the same as the grant day stock price. Thus it seems that award of executive stock option is more advantageous to the executive compared with normal executive stock option where the exercise price is equal to the grant day stock price.

Table

β (Rebate)	W(Option Value)	β (Rebate)	W(Option value)
0.05	47.26	0.45	25.33
0.10	44.52	0.50	22.59
0.15	41.78	0.55	19.85
0.20	39.04	0.60	17.11
0.25	36.30	0.65	14.37
0.30	33.55	0.70	11.63
0.35	30.61	0.75	8.89
0.40	28.07	0.80	6.14

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Appendix

We assume that the price of the underlying stock of the corporation at time t (≥ 0) is represented by S(t) and that

$$S(t) = S(0)\exp\{X(t)\}, t \geq 0, \qquad \text{(A.1)}$$

where the stochastic process {X(t)} has stationary and independent increments and is continuous in probability. Furthermore, we assume that {X(t)} has a Brownian motion with mean μ per unit time and variance σ^2 per unit time. Let the density function of {X(t)} be $f(x,t)$. Following Gerber and Shui (1994a and 1994b), we introduce a new density function given by

$$\frac{\exp(hx)f(x,t)}{E[e^{hx}]} \qquad \text{(A2)}$$

where the parameter h is called the Esscher parameter.

We determine the value of the parameter h(say,h=h^*) so that the discounted stock price is arbitrage-free. Thus, if we denote the risk-free rate as r, then we have

$$S(0) = E[e^{-rt}S(t);h^*] \qquad \text{(A.3)}$$

where expectation is under the new measure with density given in (2.2).

Suppose the underlying stock pays dividend at a rate δ per unit time.This leads to the relation which identifies the value of the Esscher parameter :

$$\mu + h^*\sigma^2 = r - \delta - \frac{\sigma^2}{2}. \qquad \text{(A.4)}$$