

PRICING EQUITY LINKED ANNUITIES UNDER REGIME-SWITCHING GENERALIZED GAMMA PROCESS

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Abstract

We propose a model for valuing equity linked annuity (ELA) products under a generalized gamma model with a Markov-switching compensator. We suppose that the market interest rate and all the parameters of the underlying reference portfolio switch over time according to the state of an economy, which is modelled by a continuous-time Markov chain. The model considered here can provide market practitioners with flexibility in modelling the dynamics of the reference portfolio. We price the ELA by pricing its embedded options, for which we employ the regime-switching version of Esscher transform to determine the pricing kernel. A system of coupled partial-differential-integral equations satisfied by the embedded option prices is derived. Simulation results of the model have been presented and discussed.

Keywords: Equity, Pricing, Gamma Model, ELA

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1 Introduction

In recent years, the insurers have increasingly been faced with market demand for a hybrid of the fixed and variable annuities leading. Fixed annuities have been attractive investments as investors are guaranteed a certain fixed rate for a period of 12 months or longer. However, in declining interest rate markets, investors may suffer from a lower yield, as it is usually tied to the fixed yield instruments. On the other hand, variable annuities can offer the potential for higher returns, however, portfolio allocation in anticipation of market changes could be a burden on investors.

Equity linked annuities (ELAs) are a popular class of equity linked insurance products around the world. In these policies the insured not only receives the guaranteed annual minimum benefit, but also receives proceeds from a reference portfolio such as S&P 500. ELAs, typically, have a cap which is an upper bound of the annual return and so the level of return is limited no matter how high is the return of the reference portfolio. The latter point, differentiates ELAs from other popular types of equity linked products, such as participating products, in the sense that participating products maintain a buffer account where the good-years' over performance is accrued to smooth the under performance of the bad-years (see [10] for further details). ELAs, not having this feature, are offered at a lower premium in the insurance market. Therefore, it could be argued that they could be more cost effective if they are linked to well-diversified portfolios and depending on the level at which the cap is set.

Accurate pricing of equity linked policies, through the fair valuation of the embedded options, can be traced back to [36]. Majority of the previous research on pricing ELAs, only consider the Black-Scholes economy, which may not capture the stylistic features of the portfolio returns. For instance, [34] derived a pricing formula for point-to-point ELAs and [24] studied ELAs with path dependent optionality. Under the same modeling framework, but with stochastic interest rate, [27] derived a pricing formula for ELAs by using the risk-minimization hedging strategy. Moreover, [22] also studied the stochastic interest rate case for the valuation of ratchet equity-indexed annuities.

In this paper, we propose a model for the valuation of ELA products under a generalized gamma model with a Markov-switching compensator. Therefore, not only will we capture the random jumps of the underlying reference portfolio, but also we model the structural changes in the economy. We make the assumption that the parameters of the market values of the reference portfolio, namely, the risk-free interest rates, the expected growth rate and the volatility of the risky asset, depend on the state of the economy, which is modeled by a continuous-time Markov-chain process. Our model is a modified version of the kernel-biased representation of [19], [20]. Incorporation of the the Markov chain process to this framework provides further flexibility to describe the impact of structural changes in macroeconomic conditions and business cycles on the valuation model. We utilize the Esscher transform to determine the equivalent martingale measure and price the participating product under the generalized jump-diffusion model.

The concept of regime-switching can be traced back to [29] [15] where they employed regime-switching regression models to describe nonlinearity in economic data. The idea of probability switching appeared in the early development of nonlinear time series analysis, where [35] proposed one of the oldest classes of nonlinear time series models, namely the threshold models. Regime-switching models aim to capture the appealing idea that the macro-economy is subject to regular, yet unpredictable in time, regimes, which in turn effect the prices of financial securities. For example, structural changes of macro-economic conditions, such as inflation and recession, may induce changes in the stock returns or in the term structure of interest rates, and similarly, periods of high market turbulence and liquidity crunches may increase the default risk of financial institutions.

[17] popularized regime-switching time series models in the economic and econometric literature and since then, considerable attention has been paid to investigate the use of regime-switching to model economic and financial data. Due to the empirical success of regime-switching models, they have been applied to different areas in banking and finance; including asset allocation, option valuation, risk management, term structure modeling. Recently, scholars have turned their attention to option valuation under regime-switching model, including, [28],[16], [2], [6], and [11]. Additionally, regime-switching models have become popular in actuarial science in recent years. For example, [18], [33], and [32] used the regime-switching models to capture the impact of the structural changes in the economy on the value of different equity linked insurance products.

2 Modeling framework

To start, suppose that (Ω, \mathcal{F}, P) is a probability space where P is probability measure. Assume also that Ω the space is filtered by a non-decreasing right continuous family \mathcal{F}_t of sub- σ -fields of \mathcal{F} . Moreover, let \mathcal{T} denote the time index that takes value on the interval $[0, T]$. We describe the states of the economy by a continuous-time Markov chain $\{X_t\}_{t \in \mathcal{T}}$ on (Γ, \mathcal{F}, P) with a finite state space $\mathcal{S} := (s_1, s_2, \dots, s_N)$. Without loss of generality, we can identify the state space of the process $\{X_t\}_{t \in \mathcal{T}}$ to be a finite set of unit vectors $\{e_1, e_2, \dots, e_N\}$, where $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^N$. From Elliott et al. [5] we present the following semi-martingale decomposition for the process $\{X_t\}_{t \in \mathcal{T}}$ is

$$X_t = X_0 + \int_0^t QX_s ds + M_t, \quad (1)$$

where Q and M_t is a \mathbb{R}^N -valued martingale with respect to the filtration generated by $\{X_t\}_{t \in \mathcal{T}}$.

Let r_t be the instantaneous market interest rate which depends on the state of the economy, and defined as follows

$$r_t := r(t, X_t) = \sum_{i=1}^N r_i \langle X_t, e_i \rangle, \quad t \in \mathcal{T}, \quad (2)$$

where $r := (r_1, r_2, \dots, r_N)$ with $r_j > 0$ for each $j = 1, 2, \dots, N$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in the space \mathbb{R}^N ; consequently, the dynamic of the value of the risk-free asset, $\{B_t\}_{t \in \mathcal{T}}$ is given by:

$$\frac{dB_t}{B_t} = r(t, X_t) dt,$$

where $B_0 = 1$. Let $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ denote a measurable space, where $\mathcal{B}(\mathcal{T})$ is the Borel σ -field generated by the open subsets of \mathcal{T} . Let \mathcal{X} denote $\mathcal{T} \times \mathbb{R}^+$, then $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is a measurable space. Let $U := \{U_t\}_{t \in \mathcal{T}}$ denote a Poisson process on (Ω, \mathcal{F}, P) , with $U_0 = 0$, P -a.s. Write $\Delta U_s = U_s - U_{s-}$. Then,

$$U_t = \sum_{0 < s \leq t} \Delta U_s.$$

A Poisson random measure $N(dz, dt) := N(dz, dt; w)$ on \mathcal{X} is a non-negative, integer-valued random measure on \mathcal{X} induced by the process U . It can be defined as follows:

$$N(A, (0, T]) := \sum_{s \in \mathcal{T}} \mathbf{1}_{\Delta U_s \neq 0} \delta_{(s, \Delta U_s)}((0, T] \times A), \quad A \subset \mathbb{R}^+.$$

Let $N_{X_t}(\cdot, U)$ denote a Markov-switching Poisson random measure on the space \mathcal{X} . Write $N_{X_t}(dt, dz)$ for the differential form of measure $N_{X_t}(t, U)$. Let $\rho_{X_t}(dz | t)$ denote a Markov-switching Levy measure on the space \mathcal{X} depending on t and the state X_t ; η is a σ -finite (nonatomic) measure on \mathcal{T} . James [19], [20] defined a kernel-biased representation of completely random measures. By this approach they have provided a great deal of insight into the different types of finite and infinite

jump modeling by choosing different kernel functions. We define the Markov-modulated version of the generalized gamma (MGG) process, using the following Markov-modulated version of the representation of completely random measures.

$$\mu_{X_t}(dt) := \int_{\mathbb{R}^+} z N_{X_t}(dt, dz).$$

By using an arbitrary positive function on \mathbb{R}^+ , z , ρ_i and η for each bounded set \mathbf{B} in \mathbf{T} , we have

$$\sum_{i=1}^N \int_{\mathbf{B}} \int_{\mathbb{R}^+} \min(z, 1) \rho_i(dz | t) \eta(dt) < \infty. \quad (3)$$

$$\nu_{X_t}(dt, dz) := \rho_{X_t}(dz | t) \eta(dt) = \frac{1}{\Gamma(1-\alpha) z^{(1+\alpha)}} \sum_{i=1}^N e^{-b_i z} \langle b, X_t \rangle dz \eta(dt). \quad (4)$$

The generalized gamma process nests a number of very important processes in finance and actuarial studies. When $\alpha = 0$, the MGG process reduces to a Markov modulated weighted gamma (MWG) process. When $\alpha = 0.5$ the MGG process becomes the Markov modulated inverse Gaussian (MIG) process. Let $\{W_t\}_{t \in \mathbf{T}}$ denote a standard Brownian motion, and $\tilde{N}_{X_t}(dt, dz)$ denote the compensated Poisson random measure and it is defined as:

$$\tilde{N}_{X_t}(dt, dz) = N_{X_t}(dt, dz) - \rho_{X_t}(dz | t) \eta(dt). \quad (5)$$

Furthermore, suppose that μ_t and σ_t denote the drift and volatility of market valuation model, where $\mu := (\mu_1, \mu_2, \dots, \mu_N)$ and

Let $\alpha \geq 0$ denote a constant shape parameter of the MGG process. We suppose that the scale parameter of the MGG process, b_t , switches over the time according to the states of the Markov chain X . Let $b := (b_1, b_2, \dots, b_N) \in \mathbb{R}^N \geq 0$ for each $i = 1, 2, \dots, N$, such that

$$b_t := \langle b, X_t \rangle = \sum_{i=1}^N b_i \langle X_t, e_i \rangle.$$

Then,

$\sigma := (\sigma_1, \sigma_2, \dots, \sigma_N)$; $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$ for each $i = 1, 2, \dots, N$, then

$$\mu_t := \langle \mu, X_t \rangle = \sum_{i=1}^N \mu_i \langle X_t, e_i \rangle,$$

$$\sigma_t := \langle \sigma, X_t \rangle = \sum_{i=1}^N \sigma_i \langle X_t, e_i \rangle. \quad (6)$$

Then, consider a generalized jump-diffusion process $A := \{A(t) | t \in \mathbf{T}\}$, such that

$$dA_t = A_{t-} [\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}^+} z \tilde{N}_{X_t}(dt, dz)], \quad (7)$$

where $A_0 = 0$. We assume under P the price process $\{S_t\}_{t \in \mathbf{T}}$ is defined as $S_t := \exp(A_t)$. Thus

$$dS_t = (\mu_t + \frac{1}{2} \sigma_t^2) dt + \sigma_t dw_t - \int_{\mathbb{R}^+} \{z - \exp(z) + 1\} \times \rho_{X_t}(dz | t) \eta(dt) + \int_{\mathbb{R}^+} (\exp(z) - 1) \tilde{N}_{X_t}(dt, dz).$$

3 Pricing by the Esscher transform

The market described in Section 4 is incomplete, consequently, there is more than one pricing kernel. Different approaches have been proposed for pricing and hedging derivative securities in incomplete financial markets. For instance, Follmer and Sondermann [12] and [30] selected an equivalent martingale measure by minimizing the quadratic utility of the terminal hedging errors. Davis [4] adopted an economic approach based on the marginal

rate of substitution to pick a pricing measure via a utility maximization problem. Avellaneda [1], Frittelli [13], and Fard and Siu [11] employed the minimum entropy martingale measure method to choose the equivalent martingale measure.

Here, we employ a time-honoured tool in actuarial science, namely, the Esscher transform, to select an equivalent martingale measure. The Esscher transform was first introduced to the actuarial science literature by Escher [9], where it was used to approximate the distribution of aggregate claims. It

was also adopted to premium calculation. Gerber and Shiu [14] pioneered the Esscher transformation approach for option valuation. They provided an economic equilibrium justification for a price of an option determined by the Esscher transform based on the maximization of an expected power utility of an economic agent. Elliott and Kopp [7] demonstrated that the Esscher transform is consistent with the minimal entropy martingale measure (MEMM) approach for option valuation. In what follows, we adopt a version of the Esscher transform for general semi-martingales in Kallsen and Shiryaev [21] and Elliott and Siu [8].

Let $\mathbf{F}^X := \{\mathbf{F}_t^X\}_{t \in \mathbb{T}}$, $\mathbf{F}^A := \{\mathbf{F}_t^A\}_{t \in \mathbb{T}}$ and $\mathbf{F}^S := \{\mathbf{F}_t^S\}_{t \in \mathbb{T}}$ denote the P -augmentation of the natural filtration generated by A and S , respectively. Since, \mathbf{F}^A and \mathbf{F}^S are equivalent, we can use either one of them as an observed information structure. Hence, define \mathbf{G}_t for the σ -algebra

$\mathbf{F}^X \vee \mathbf{F}^A$ for each $t \in \mathbb{T}$. Further, let $L(A)$ be the space of processes $\theta := \{\theta(t) \mid t \in \mathbb{T}\}$ such that

$$(\theta.A)_t := \int_0^t \theta(u) dA(u).$$

This is the stochastic integral of θ with respect to A . Let $\{\Lambda_t\}_{t \in \mathbb{T}}$ denote a \mathbf{G} -adapted stochastic process defined as below:

$$\Lambda_t := \frac{e^{(\theta.A)_t}}{\mathbf{M}(\theta)_t}, \quad t \in \mathbb{T}. \quad (8)$$

Where $\mathbf{M}(\theta)_t := E^P[e^{(\theta.A)_t} \mid \mathbf{F}_t^A]$, is a Laplace cumulant process. Apply Ito's differentiation rule for jump diffusion

$$\begin{aligned} e^{(\theta.A)_t} &= 1 + \int_0^t e^{(\theta.A)_s} \theta_s \left(\mu_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^t e^{(\theta.A)_s} \theta_s \sigma_s dW_s \\ &\quad + \int_0^t \int_{\mathbb{R}^+} e^{(\theta.A)_s^-} (e^{\theta_s z} - 1 - \theta_s z) \rho_{X_t}(dz \mid s) \eta(ds) \\ &\quad + \int_0^t \int_{\mathbb{R}^+} e^{(\theta.A)_s^-} (e^{\theta_s z} - 1) \tilde{N}_{X_s}(ds, dz) \end{aligned} \quad (9)$$

Therefore

$$\begin{aligned} \Lambda_t &= \exp \left[\int_0^t \theta_s \sigma_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 \sigma_s^2 ds + \int_0^t \int_{\mathbb{R}^+} \theta_s z \tilde{N}_{X_s}(dz, ds) \right. \\ &\quad \left. - \int_0^t \int_{\mathbb{R}^+} (e^{\theta_s z} - 1 + \theta_s z) \rho_{X_s}(dz \mid s) \eta(ds) \right]. \end{aligned} \quad (10)$$

Lemma 3.1 Λ_t is P martingale w.r.t \mathbf{G}_t .

Proof. James [19], [20] showed that

$$E[\exp(\int_0^t \int_{\mathbb{R}^+} \theta_s z \tilde{N}_{X_s}(ds, ds)) \mid \mathbf{G}_t] = \exp(\int_0^t \int_{\mathbb{R}^+} (e^{\theta_s z} - 1 + \theta_s z) \rho_{X_s}(dz \mid s) \eta(ds))$$

Then, by taking the conditional expectations of (10), the results follow.

Further,

$$\Lambda_t = 1 - \int_0^t \Lambda_s \theta_s \sigma_s dW_s + \int_0^t \int_{\mathbb{R}^+} \Lambda_s (e^{\theta_s z} - 1) \tilde{N}_{X_t}(dz, ds) \quad (11)$$

Then for each $\theta \in L(A)$, we define a new probability measure $P^\theta : P$ on $\mathbf{G}(T)$ by the Radon-Nikodym derivative:

$$\frac{dP^\theta}{dP} \Big|_{\mathbf{G}(T)} := \Lambda_T. \quad (12)$$

This new measure P^θ is defined by the Esscher transform Λ_T associated with $\theta \in L(A)$. According to the fundamental theorem of asset pricing, the absence of arbitrage essentially means there exist an equivalent martingale measure under which discounted asset prices are local-martingales; which is widely known as the local-martingale condition.

Now we stipulate a necessary and sufficient condition for the local martingale condition.

Proposition 3.2. For each $t \in \mathbb{T}$, let the discounted price of the risky asset at time t be $\tilde{S}(t) = e^{-rt} S(t)$ Then the discounted price process

$\tilde{S} := \{\tilde{S}(t) \mid t \in \mathbf{T}\}$ is an P^θ -local-martingale if and only if:

$$\theta_t := \langle \theta, X_t \rangle \quad t \in \mathbf{T},$$

is such that $\theta := (\theta_1, \theta_2, \dots, \theta_N) \in \mathbf{R}^N$ satisfies the following equation:

$$\mu_t - r_t + \theta_t \sigma_t^2 + \int_{\mathbf{R}^+} \{e^{\theta z} (e^z - 1) - z\} \rho_{X_t} d(z \mid t) \eta'(t) = 0 \quad (13)$$

$$\begin{aligned} E^\theta[\exp(-\int_0^t r_s ds) S_t \mid \mathbf{G}_0] &= \exp(-\int_0^t r_s ds) E[\Lambda_t \exp(\int_0^t dA_u) \mid \mathbf{G}_0] \\ &= \exp(-\int_0^t r_s ds) \frac{M(\theta+1)_t}{M(\theta)_t} \\ &= \exp(\int_0^t (\mu_s - r_s - \frac{1}{2} \sigma_s^2) ds + \frac{1}{2} \int_0^t (2\theta+1) \sigma_s^2 ds \\ &\quad + \int_0^t \int_{\mathbf{R}^+} \{e^{\theta z} (e^z - 1) - z\} \rho_{X_t} d(z \mid t) \eta(dt)). \end{aligned}$$

Then by setting time $s = 0$, and applying the martingale condition we achieve:

$$\begin{aligned} &\int_0^t (\mu_s - r_s - \frac{1}{2} \sigma_s^2) ds + \frac{1}{2} \int_0^t (1 + 2\theta) \sigma_s^2 ds \\ &+ \int_0^t \int_{\mathbf{R}^+} \{e^{\theta z} (e^z - 1) - z\} \rho_{X_t} d(z \mid t) \eta'(s) ds = 0 \end{aligned}$$

$$\begin{aligned} dA_t &= (\mu_t + 2\theta \sigma_t^2 - \frac{1}{2} \sigma_t^2) dt + \int_{\mathbf{R}^+} z(1 - e^{-\theta z}) \rho_{X_t}^\theta(dz \mid t) \eta(dt) + \sigma_t d\tilde{W}_t \\ &\quad + \int_{\mathbf{R}^+} z \tilde{N}_{X_t}^\theta(dz, dt). \end{aligned} \quad (14)$$

Proof. Assume that $P: P^\theta$ with density process Λ_t . Note that θ is a deterministic process, satisfying

$$\int_0^t (\sigma_s \cdot \theta^2) ds < \infty,$$

Further suppose $Z_u \in BM(\mathbf{T})$. Then by a version of Bayes' rule

$$\begin{aligned} M_A^\theta(Z)_t &:= E^\theta[e^{(Z, A)_t} \mid \mathbf{G}_0] = E[\Lambda_t e^{(Z, A)_t} \mid \mathbf{G}_0] \\ &= \exp(\int_0^t Z(\mu_s - \frac{1}{2} \sigma_s^2) ds + \int_0^t \frac{1}{2} (Z + \theta)^2 \sigma_s^2 ds \\ &\quad + \int_0^t \int_{\mathbf{R}^+} \{e^{(Z+\theta)z} - 1 - (Z+\theta)z\} \rho_{X_t}(dz \mid s) \eta(ds) \\ &\quad - \int_0^t \frac{1}{2} (\theta \cdot \sigma_s)^2 ds - \int_0^t \int_{\mathbf{R}^+} \{e^{\theta z} - 1 - \theta z\} \rho_{X_t}(dz \mid s) \eta(ds)) \\ &= \exp(\int_0^t Z(\mu_s + 2\theta \sigma_s^2 - \frac{1}{2} \sigma_s^2) ds \\ &\quad + \int_0^t \int_{\mathbf{R}^+} (e^{\theta z} - 1) \rho_{X_t}(dz \mid s) \eta(ds) + \frac{1}{2} \int_0^t Z^2 \sigma_s^2 ds \\ &\quad + \int_0^t \int_{\mathbf{R}^+} \{e^{Zz} - 1 - Zz\} e^{\theta z} \rho_{X_t}(dz \mid s) \eta(ds)). \end{aligned}$$

Proof. Since \tilde{S} is \mathbf{F}^A -adapted, \tilde{S} is an (\mathbf{F}^Y, P^θ) -local-martingale if and only if it is a (\mathbf{G}, P^θ) -local-martingale. By Lemma 7.2.2 in Elliott and Kopp [7], \tilde{S} is an (\mathbf{G}^Y, P^θ) -local-martingale iff $\Lambda \tilde{S} := \{\Lambda(t) \tilde{S}(t) \mid t \in \mathbf{T}\}$ is a (\mathbf{G}, P^θ) -local-martingale. First, by Bayes' rule:

Then under P^θ (14) holds.

Proposition 3.4 The price process of the reference portfolio S under P^θ is

$$dS_t = (r_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t d\tilde{W}_t + \int_{\mathbb{R}^+} (e^{\theta z} (e^z - 1) - z) \tilde{N}_{X_t}^\theta(dt, dz), \quad (15)$$

Proof. Recall $S_t := \exp(A_t)$. The proof can easily follow by applying Ito lemma and the martingale condition (13) to (14).

4 Point-to-Point ELA

In this section, we concentrate on pricing of ELAs using the framework presented in the previous section. It is well known that the plain vanilla point to point ELA can be assessed by separating it into the European vanilla call option. (For more details see Sheldon Lin and Tan [31], Choi and Kim [3]). The payoff value for the plain vanilla point to point ELA driven by the following equation:

$$L(t) = \max \left[\min \left(1 + \omega R_t (1 + \zeta)^t, \beta(1 + g)^t \right) \right] \quad (16)$$

where α is the participation rate, R_t measures the appreciation of the index up to time t , ζ is ceiling rate, g is floor rate and β is the percentage of unite for which the floor is applied. With particular focus on the plain vanilla point to point ELA, R_t is defined by

$$R_t = \frac{S_t}{S_0} - 1, \quad (17)$$

where S_t is the price of an equity index at time t . We can also find an alternative expression for $L(t)$ using (17), $L(t)$ is separated into two parts

$$L(t) = \beta(1 + g)^t + \frac{\omega}{S_0} (\max[S_t - K_g, 0] - \max[S_t - K_\zeta, 0]), \quad (18)$$

where K_g and K_ζ are defined as

$$K_g = \frac{S_0}{\omega} (\beta(1 + g)^t - 1 + \omega)$$

$$K_\zeta = \frac{S_0}{\omega} (\beta(1 + \zeta)^t - 1 + \omega)$$

The two maximum functions in (18) resemble the pay-off of two European vanilla call options. Therefore, we can price $L(t)$ via pricing the embedded options. More precisely, the value of a T -year maturity point to point ELA, denote by the following argument:

$$L(S_0, 0, T, \omega, \zeta, \beta, g, \sigma) = \beta \exp(-rT)(1 + g)^T + \frac{\omega}{S_0} [C(0, S_0, \sigma; T, K_g) - C(0, S_0, \sigma; T, K_\zeta)] \quad (19)$$

where $C(0, S_0, \sigma; T, K_{(\cdot)})$ denotes the price of a European call option with strike price $K_{(\cdot)}$ at expiry T . In the next section, we provide the pricing of C , using the partial integro-differential equation PIDE approach.

5 PIDE method for European options

With a slight abuse of notation, let $C(S_T) := C(0, S_0, \sigma; T, K_{(\cdot)})$. This allows us to simplify the notation. Following the risk-neutral argument, a European option is valued by discounting the conditional expectation of the terminal payoff under the risk-neutral martingale measure. Then, given $S_t = S$ and $X_t = X$,

$$\tilde{C}(S_T) := E^\theta [\exp(-\int_0^T r_u du) C(S_T) | S_t = S, X_t = X]. \quad (20)$$

Recall that the stock price dynamic under P^θ is given by

$$dS_t = (r_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t d\tilde{W}_t + \int_{\mathbb{R}^+} (e^{\theta z} (e^z - 1) - z) \tilde{N}_{X_t}^\theta(dt, dz).$$

Let $\tilde{C}_i = \tilde{C}(t, T, S, e_i)$ for $i = 1, 2, \dots, n$ and $\tilde{C} := (\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_N)$ then, Then, by Itô's differentiation rule for semi-martingales,

$$\begin{aligned}
\tilde{C}(t, T, S, X) &= \tilde{C}(0, T, S_0, X_0) + \int_0^t \frac{\partial \tilde{C}}{\partial u} du \\
&+ \int_0^t \frac{\partial \tilde{C}}{\partial S} S_{u-} \left((r_{u-} - \frac{1}{2} \sigma_{u-}^2) + \sigma_{u-} d\tilde{W}_u \right) + \frac{1}{2} \int_0^t \sigma_{u-}^2 S_{u-}^2 \frac{\partial^2 \tilde{C}}{\partial S^2} du \\
&+ \int_0^t \int_{\mathbb{R}^+} [\tilde{C}(u, T, S_{u-} + (e^{\theta_z}(e^z - 1) - z), X_{u-}) \\
&- \tilde{C}(u, T, S_{u-}, X_{u-}) - \frac{\partial \tilde{C}}{\partial S} (e^{\theta_z}(e^z - 1) - z)] v(du, dz) \\
&+ \int_0^t \int_{\mathbb{R}^+} [\tilde{C}(u, T, S_{u-} + (e^{\theta_z}(e^z - 1) - z), X_{u-}) \\
&- \tilde{C}(u, S_{u-})] dN(du, dz) + \int_0^t \langle C, AX_{u-} du + dM_u \rangle.
\end{aligned}$$

Then by setting the bounded variation terms to zero, we obtain the following PIDE, the solution of which is the European option price C .

$$\begin{aligned}
&\frac{\partial \tilde{C}}{\partial t} + \frac{\partial \tilde{C}}{\partial S} S_{t-} \left(r_{t-} - \frac{1}{2} \sigma_{t-}^2 \right) + \frac{1}{2} \sigma_{t-}^2 S_{t-}^2 \frac{\partial^2 \tilde{C}}{\partial S^2} \\
&+ \int_{\mathbb{R}^+} [\tilde{C}(t, T, S_{t-} + (e^{\theta_z}(e^z - 1) - z), X_{t-}) - \tilde{C}(t, T, S_{t-}, X_{t-}) \\
&- \frac{\partial \tilde{C}}{\partial S} (e^{\theta_z}(e^z - 1) - z)] v(dt, dz) + \langle \tilde{C}, AX_{t-} \rangle = 0
\end{aligned}$$

Let $C_i := C(t, T, s, e_i)$, for each $i = 1, 2, \dots, N$ and $C := (C_1, C_2, \dots, C_N)$. Then, the European option price C satisfies the following regime-switching PIDE:

$$\begin{aligned}
r_t C &= \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} S_{t-} \left(r_{t-} - \frac{1}{2} \sigma_{t-}^2 \right) + \frac{1}{2} \sigma_{t-}^2 S_{t-}^2 \frac{\partial^2 C}{\partial S^2} \\
&+ \int_{\mathbb{R}^+} [C(t, T, S_{t-} + e^{\theta_z}(e^z - 1) - z, X_{t-}) - C(t, T, S_{t-}, X_{t-}) \\
&- \frac{\partial C}{\partial S} (e^{\theta_z}(e^z - 1) - z)] v(dt, dz) + \langle C, AX_{t-} \rangle = 0
\end{aligned}$$

with the terminal condition $C(t, T, S, X) = C(S_T)$.

Then, \tilde{C} satisfies the following n -coupled PIDE:

$$\begin{aligned}
r_t C &= \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} S_{t-} \left(r_{t-} - \frac{1}{2} \sigma_{t-}^2 \right) + \frac{1}{2} \sigma_{t-}^2 S_{t-}^2 \frac{\partial^2 C}{\partial S^2} \\
&+ \int_{\mathbb{R}^+} [\tilde{C}(t, T, S_{t-} + e^{\theta_z}(e^z - 1) - z, e_i) - C(t, T, S_{t-}, e_i)]
\end{aligned}$$

$$- \frac{\partial C}{\partial S} (e^{\theta_z}(e^z - 1) - z)] v(dt, dz) + \langle C, Ae_i \rangle = 0$$

with the terminal condition $C(t, T, S, e_i) = C(S_T)$.

6 Simulation experiments

In this section, we conduct simulation experiments to compare the price equity linked annuities implied by various parametric specifications of our generalized gamma model. We highlight some features of the qualitative behavior of the fair values of the options that can be obtained from different parametric specifications of our model.

We adopt the Poisson weighted algorithm by [25] to simulate completely random measures. The Poisson weighted algorithm is applicable for a wide class of completely random measures, which are very difficult to simulate directly. The main idea of the Poisson weighted algorithm is that instead of generating jump sizes of a completely random measure directly from a nonstandard density function, one can first generate jump sizes from a proposed density function, i.e. the gamma density, and then adjust the simulated jump sizes by the corresponding Poisson weights. The Poisson weights are simulated from a Poisson distribution with intensity parameter given by the odd ratio of the compensator of the completely random measure and the compensator corresponding to the proposed density.

Assume that the time interval to maturity $[0, T]$ is divided into n subintervals with equal length of

$$\Delta t := \frac{T}{n} \quad \text{and} \quad j = 1, \dots, n-1.$$

Moreover, Let consider that M denote the number of jumps of the completely random measure over the time horizon $[0, T]$. It is necessary to mentioned that we let $M \ll n$. Here, we set $n = 1,000$ and $M = 200$.

We generate 10,000 simulation path for $\{X_j\}_{j=1, \dots, 1000}$ over the 10 years. The process generated by the Poisson weighting algorithm

converges in distribution to the completely random measure $\mu(t)$ on the space $D[0, T]$ under the Skorohod topology for sufficiently large N (for the proof see [26]). Next, we consider two state Markov chain process X , the first or base regime ($N = 1$) describes the 'stable economic state' price behavior and the second regime ($N = 2$) represents the 'volatile economic state'. Furthermore, the transition probability matrix is supposed as follows:

$$\begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Next, we utilize the popular forward Euler discretization scheme to approximate the paths of the continuous time process, when performing the simulation. [23] showed that under some conditions, it can be shown that the Euler approximation converges weakly to the target continuous-time process when the number of discretization intervals tends to infinity.

For the parameters of the vulnerable option, we assume the specimen values in Table 1.

Table 1. Model parameter values

Parameter name	Value
Initial price of the reference portfolio	$S_0 = 100$
Term to Maturity	$T = 10$ year
Volatility of underlying asset in Regime 1	$\sigma_1 = 0.25$
Volatility of underlying asset in Regime 2	$\sigma_2 = 0.35$
Drift of underlying asset in Regime 1	$\mu_1 = 0.10$
Drift of underlying asset in Regime 2	$\mu_2 = 0.05$
Interest rate in Regime 1	$r_1 = 0.06$
Interest rate in Regime 2	$r_2 = 0.11$
Participation rate	$\alpha = 0.75$
Percentage of unit for which the minimum guarantee is applied	$\beta = 0.8$

We suppose that the shape parameter α for the MGG processes from 0.0 to 0.9, with an increment of 0.1. When $\alpha = 0.0$, the MGG process becomes the MWG process. When $\alpha = 0.5$, the MGG process becomes the MIG process. Other values of α generate different parametric forms of the MGG processes. We assume that the parameter values of the no-regime-switching versions of these processes match with those in the corresponding regime-switching processes when the economy is in "State 1." In all figures, "with Markov switching" refers to the models with both the GG component and the model parameters being modulated by the two-state Markov chain; "without Markov switching" refers to the models with the jump component and constant model parameters.

Additionally, we consider the scale-distorted version of the MGG and GG processes. We focus on investigating the impact of different values of the scale distortion parameter b on the fair values of the policy. In particular, we suppose that b takes values 0.5, 1.0 (i.e., no scale distortion), 2.0, 2.5 and 3.0. Throughout this subsection, we suppose that the shape parameter $\alpha = 0.5$ for the scale-distorted version of the MGG and GG processes.

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Appendix A

Figure A.1. The fair values of ELA policies for different shape parameters

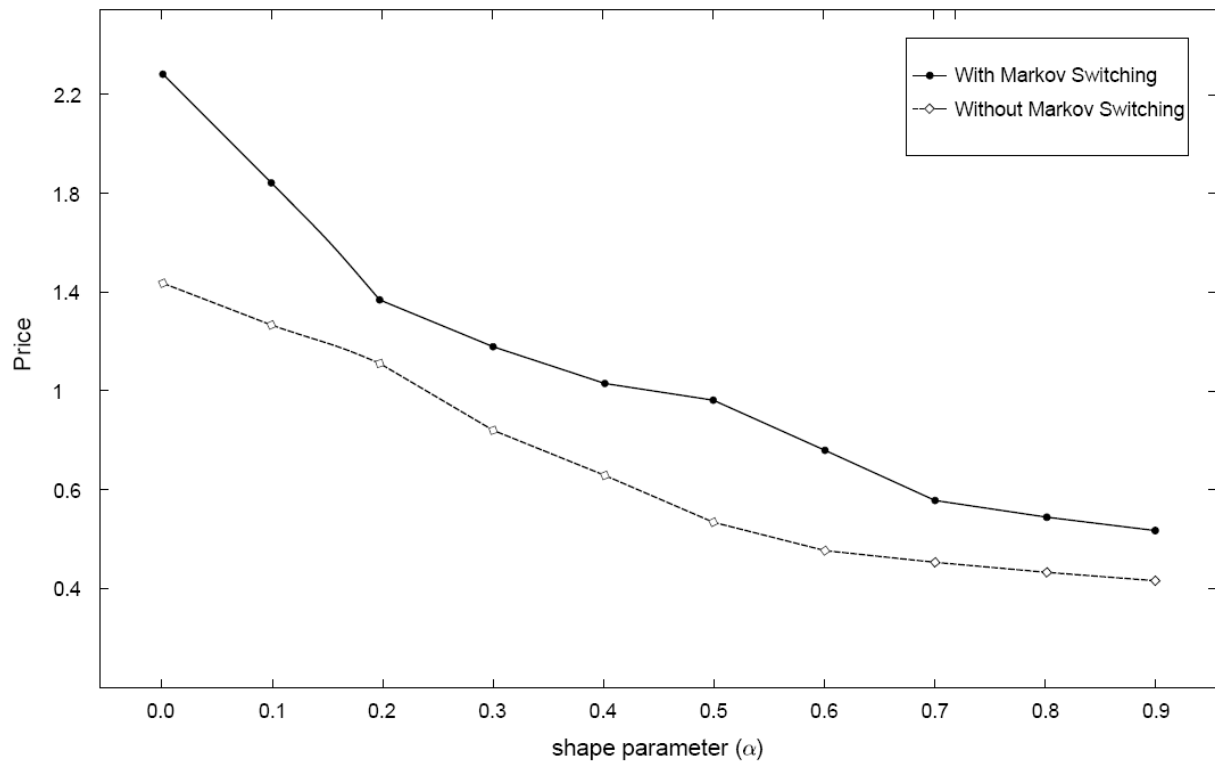


Figure A.2. The fair values of ELA policies for different scale parameters

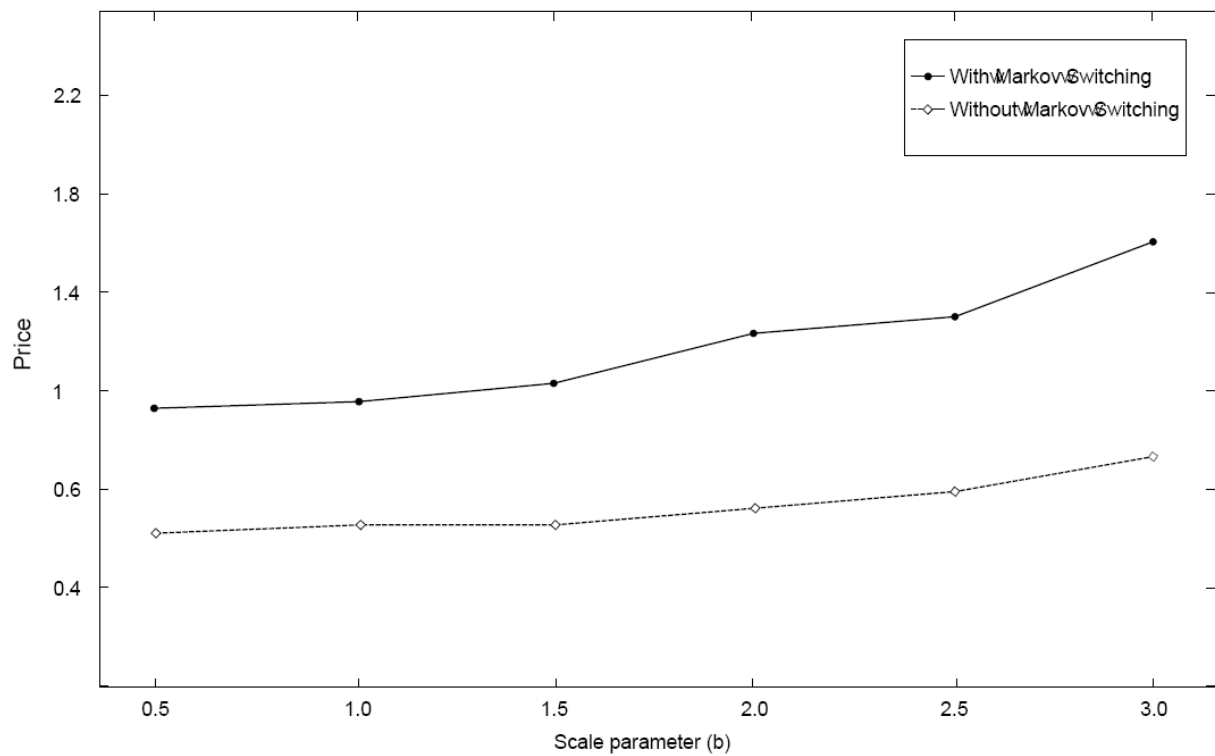


Figure A.3. The term structure for different ELA prices

