MIXTURE OF PROBABILISTIC FACTOR ANALYZERS FOR MARKET RISK MEASUREMENT: EMPIRICAL EVIDENCE FROM THE TUNISIAN FOREIGN EXCHANGE MARKET

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How to cite this paper:
Mosbahi, M. N., Saidane, M., Messabeb, S. (2017). Mixture Of Probabilistic Factor Analyzers For Market Risk Measurement: Empirical Evidence From The Tunisian Foreign Exchange Market. *Risk governance & control: financial markets & institutions, 7(2,1), 158-169.*
http://dx.doi.org/10.22495/rgcv7i2c1p4

ISSN Online: 2077-4303
ISSN Print: 2077-429X

Received: 05.02.2016
Accepted: 23.03.2017

JEL Classification: C38, C63, F3, G17, G31
DOI: 10.22495/rgcv7i2c1p4

Abstract

In this paper, we propose a new approach for Basel-Compliant Value-at-Risk (VaR) estimation in financial portfolio risk management, which combines Gaussian Mixture Models with probabilistic factor analysis models. This new mixed specification provides an alternative, compact, model to handle co-movements, heterogeneity and intra-frame correlations in financial data. This results in a model which concurrently performs clustering and dimensionality reduction, and can be considered as a reduced dimension mixture of probabilistic factor analyzers. For maximum likelihood estimation we have used an iterative approach based on the Alternating Expectation Conditional Maximization (AECM) algorithm. Using a set of historical data in a rolling time window, from the Tunisian foreign exchange market, the model structure as well as its parameters are determined and estimated. Then, the fitted model combined with a modified Monte-Carlo simulation algorithm was used to predict the VaR. Through a Backtesting analysis, we found that this new specification exhibits a good fit to the data compared to other competing approaches, improves the accuracy of VaR prediction, possesses more flexibility, and can avoid serious violations when a financial crisis occurs.

Keywords: Value-at-Risk, Gaussian Mixture Model, Latent Factor Model, Mixture of Factor Analyzers, AECM Algorithm, Monte Carlo simulation, FX Market

1. INTRODUCTION

Value at Risk has received much attention in recent years due to its importance of the link between capital and the amount of risk that the financial institution can tolerate; it does not derive from supervisors' constraints only (Basel Committee on Banking Supervision, BCBS) but is instead on the central of the risk management. In addition, Value at Risk (VaR) is a standard tool used to evaluate market risk and to estimate future losses on a portfolio of financial assets (or a single financial asset) with a confidence interval and during a period of time (Corkalo, 2011). According to this unit of measurement, level of losses in the exceptional periods is larger than normal periods; also, it facilitates the market risk measurement by a single number easy to interpret (see Linsmeir and Pearson, 1996; Stavroyiannis and Zarganas, 2013).

Consequently, much research has focused on required economic capital as an internal estimate of the capital financial institution needs, using different approaches. The main difficulty is to produce a correct and effective "VaR implementation", then to adopt internal and more sophisticated models, under certain constraints and after supervisory validation, with the expectation that valid approaches could propose, on a regular, a lower capital charge than basic ones. BCBS (2009) indicates that to evaluate market risk using their internal models, banks should be able to manipulate sophisticated models in many areas such as: risk supervision, audit control, etc. Many studied suggested that a potential cause of increasing the risk of loss is resulting from inadequate systems (as failed internal processes) or from external events.

It appears that actual regulations and standard approaches for estimating the VaR, based mainly on the Normal assumption, have been invalidated by many studies (see Berkowitz and O'Brien, 2002; Carol and Sheedy, 2008) as they strongly underestimate the extreme events observed in the market. The successive financial crises led to a greater attention to modelling the tail behaviour of the induced returns distributions, and to the use of probabilistic mixture of factor analyzers as a central concept in risk management.

The purpose of this research is to apply the value at risk methodologies using the Gaussian Mixture model (GMM), the probabilistic Factor Analysis model (FA) and our proposed Mixture of Factor Analyzers model (MFA), and to discuss their implications and consequences on the Tunisian FX decision-making process. This article deals with risk, capital requirement, and the relationship between the two, the intent to link VaR measurement methodologies with their impact on internal processes for the Tunisian FX Market, therefore, the VaR numbers generated by them are also different. The selection of an appropriate method to estimate VaR is thus difficult and complicated at the same time. Hence it becomes necessary to Backtest these VaR methods for exceptions in order to judge their performance for the Tunisian FX Market over a time period. This should give us some ideas of which method would be able to better satisfy the Tunisian FX market needs.

The paper is structured as follows: After the introduction, we present the specificity of the Tunisian FX Market. Section 3 deals with the empirical literature and discusses the three approaches that can be used to evaluate the performance of a VaR model using the Backtesting models. In section 4, we introduce our dataset and debate the statistic characteristics. Section 5 discusses the performance of a VaR model such testing and the consequence that we draw from the study over the FX Market. Perspectives and conclusions are then summarized in the last section.

1. THE CHARACTERISTIC OF THE TUNISIAN FOREIGN EXCHANGE MARKET

Nowadays, the fluctuation of a country's exchange rates depends mainly on its macro-economic indicators and on its financial stability (Samson, 2013; Aron et al., 2014). For that reason, an assessment of risks related to exchange rates is required. The first objective of currency risk management is to take into account the negative effect of daily exchange rate volatility. In addition, managing foreign exchange risks presents one of the most significant and persistent problems for Tunisian financial institutions. Then, the choice of an exchange rate regime is great importance; it calls into question the economic policy of a country, its room for maneuver and its mode of macroeconomic adjustment. The choice of FX regime is a characteristic behaviour of a system, which maintained and adopted by mutually reinforced processes, as the absence of adequate foreign-exchange rate implementation, is one of the fundamental factors that have led to major financial losses among intuition in many countries.

The Tunisian exchange rate policy is very active; it is found that Tunisia had from 1990 till 1999 a "crawling peg exchange rate regime". Since 2000, following the IMF recommendation, the Tunisia central bank has reduced its intervention in the FX market and permitted for more flexibility in the exchange rates by applying a managed float regime. Indeed, Tunisian adopted "managed floating with no-predetermined path" for the exchange rate from 2000 till 2001, to be changed again to a "crawling peg" from 2002 till 2004. From 2005 till Mai 2007 Tunisia returned to Managed floating regime with no-predetermined path for the exchange rate, and in 2008 it adopted "Conventional pegged arrangement against a composite" (IMF). These different regimes were considered as a transitional step to a free-floating exchange rate regime. Indeed, from 2009 till 2016 the official exchange rate regime applied in Tunisia consists of a "managed Floating exchange rate regime". The Tunisian exchange rate is, according to the International Monetary Fund, more flexible, but not sustained. Many international institutions, including the IMF, supporter greater flexibility in the exchange rate to reduce tensions on reserves. However, the difficulty is that the Tunisian dinar has become more volatile despite the intervention of the Central Bank on the foreign exchange market to avoid a more pronounced depreciation. It is true that the exchange rates of the Tunisian dinars were determined by the interbank market. In this market, commercial banks, including offshore banks conducted their transactions using free negotiated rates for their resident and non-resident clients. No limit is set for the difference between the Bid-Ask spread. The Central Bank of Tunisia (BCT) intervened in the market and published bank notes as indicative, at the latest on the next day, the exchange rate for interbank foreign exchange. In other words, the dinar exchange rate is supported by the central bank's interventions in the foreign exchange market to avoid a more pronounced depreciation. The "managed Floating exchange rate regime" has led to more volatility and persistence of shocks, then to understand the implications of VaR risk modeling, it is interesting to provide specific responses to the way of managing FX risk. The comprehension of these conditions requires a corporate structure that encourages precise assessments of foreign exchange risk exposure on one hand and the application of successful foreign exchange trading activities on the other. As a result, in this paper, we provide exchange risk management methodologies based on VaR by applying three different approaches (the GMM, the FA and the MFA models), which can be improved to the specificity of the Tunisian FX perspective. Therefore, we compare the performance of VaR models in order to identify the best VaR, as practiced on a daily basis by major international banks, dealers and brokers.

3. EMPIRICAL LITERATURE AND METHODOLOGY

Many theoretical as well as empirical investigated the application of VaR using different classical and recent approaches (Parametric Approaches, Nonparametric approaches and Monte Carlo Approach…) to evaluate the FX rate risk (for instance, Tokmakçioglu, 2009; Akbar and Chauveau, 2009; Ben Rejeb et al, 2012; Akhtekhane and Mohammadi, 2012; Fiksriyoso and Surya, 2013; Batten et al, 2014; Salhi et al, 2016). As can be seen, the classical VaR measures, assumed that the return distribution of financial factor risk is normally distributed, thus, the skewness and heavy tail are the two important characteristics of observed fluctuation FX Market. Provoked by these findings and in order to illustrate more light on this issue and to address the significant heterogeneity observed across the FX rates, we depart from the preview literature and employ the mixture of factor analysers as modelling dependency for the risk management. Factor models have been established for various reasons: first, these models are understandable and simple to manipulate (The fault times are independent conditionally to a random factor), second they respect the financial intuition (the dependence result of a non-mutualist systemic risk) and finally they take careful into consideration the dependency of parameters risk factors. The market risk factors in this research are the most representative currencies of the Tunisian foreign debt, namely: TND versus USD, TND versus EUR and TND versus JPY. Next, the VaR is estimate with three different VaR approaches (GMM, FA and MFA) and three confidence levels (1%, 2% and 5%).

1- VaR models: Parameter estimates

The last decade, as we can see above, several empirical studies have been developed comparing different modelling methods to identify how to evaluate the VaR. In all these studies, the basic step to make a VaR measurement for a portfolio of assets, was the reconstruction of the returns probability distribution during the holding period with a confidence level. Then, the first step in this study is to calculate the returns of the Tunisian exchange rates as follows:

$$R_t = \ln p_t - \ln p_{t-1} \approx \frac{p_t - p_{t-1}}{p_{t-1}}$$

where p_t is the daily closing exchange rate at time t. This quantity can be seen as the logarithm of the geometric growth and is known in finance as continuous compounded returns.

According to Jorion (2007), VaR is defined as: “a method of assessing risk that uses standard statistical techniques used routinely in other technical fields. Loosely VaR summarizes the worst loss over a target horizon that will not be exceeded with a given level of confidence.” So the basic parameters of this unit measurement are confidence level and holding period. The most important advantage of VaR is being prepared for potential negative outcomes and so avoids them.

Mathematically, Jorion (2007) expressed VaR as

$$p(L > VaR) \leq 1 - \alpha$$

where L is the set of losses and confidence level $\alpha \in (0,1)$.

To illustrate the dynamic nature of the VaR in the Tunisian context, we used the following statistical models:

a.Latent Factor model

Factor analysis is a statistical method for modelling the covariance structure of high dimensional data using a small number of unobservable or latent variables (McLachlan and Peel, 2000). It can be described by the following generative model:

$$y_t = Af_t + \varepsilon_t$$

where $\forall\, t = 1, \dots, T$,

$$f_t \sim N(0\,,\, I_k)$$

is a collection of k common latent factors (k-dimensional state vector) and y_t is a q-dimensional observation vector. The covariance structure is captured by the factor loading matrix$A$. The mean of the observations is determined by the vector of specific or idiosyncratic factors modelled as a multivariate normal with mean vector θ and a diagonal covariance matrixΨ:

$$\varepsilon_t \sim N(\theta\,, \Psi)\,;\ \ \forall\, t = 1, \dots, T$$

The observation process is expressed as a conditional likelihood, and is given by:

$$p(y_t | f_t) = N(\theta + Af_t\,, \Psi)$$

In addition, the observation likelihood is multivariate normal with mean vector θ and covariance matrix:

$$\Sigma = AA' + \Psi$$

To obtain maximum likelihood and then estimate model parameters $\Theta = \{\theta, A, \Psi\}$ we use the iterative EM algorithm (see Dempster et al., 1977 and Rubin and Thayer, 1982).

b.Gaussian Mixture model

The aim of the GMM is to specify the future distribution of returns by a mixture of Gaussian densities. The latter is the sum of several Gaussian densities; each one has its own parameters μ and Σ. The probability density function for the GMM is defined as:

$$p(y_t|\Theta) = \sum_{j=1}^{M} c_j\, p(y_t|\mu_j, \Sigma_j)$$

and

$$p(y_t|\mu_j, \Sigma_j) = \frac{1}{(2\pi)^{q/2}|\Sigma_j|^{1/2}} exp\left\{-\frac{1}{2}(y_t - \mu_j)'\Sigma_j^{-1}(y_t - \mu_j)\right\}$$

where, M is the number of Gaussian components, q is the dimensionality of the vector y_t, c_j are the mixture weights, μ_j mean vectors and Σ_j the covariance matrices. The mixture weights must satisfy $c_j \geq 0$ and $\sum_{j=1}^{M} c_j = 1$ to make $p(y_t|\Theta)$ a valid probability density function.

Following this model, it is possible to maximise the likelihood of the complete data using the Expectation-Maximization (EM) algorithm (Dempster et al., 1977).

In the expectation step, we will identify the conditional expectation of the complete log-likelihood function. However, in the (i)-th iteration of the Maximization Step, the conditional expectation of the complete log-likelihood function is maximized through a set of parameters$\widehat{\Theta}^{(i)} = \{\hat{c}_j, \hat{\mu}_j, \hat{\Sigma}_j\}$. These parameters will be used as the old parameters set in the following iteration$(i + 1)$.

c. Mixture of Factor Analyzers

Mixture of factor analyzers represents a finite mixture of linear submodels for the distribution of the vector of observed data y_t given the latent factorsf_t. According to this approach, modelling the distribution of the observed data can be made as follows: $y_t = A_j f_{jt} + \varepsilon_{jt}$

with prob. π_j $(j = 1, \dots, M)$ (1)

for $t = 1, \dots, T$, where f_{jt} is a k-dimensional $(k < q)$ vector of latent variables called factors and A is a $q \times k$ matrix of factor loadings (parameters). The vector of common latent factors f_{jt} is distributed $N(0, I_k)$, independently of the vector of specific or idiosyncratic factorsε_{jt}, which is distributed $N(\theta_j, \Psi_j)$, where Ψ_j is a diagonal matrix $(j = 1, \dots, M)$ and whereI_k, denotes the $(k \times k)$ identity. Thus, the MFA is given by

$$p(y_t; \theta_j, \Sigma_j) = \sum_{j=1}^{M} \pi_j N(y_t; \theta_j, \Sigma_j) \quad (2)$$

where the j-th component-covariance matrix Σ_j has the form $\Sigma_j = A_j A_j' + \Psi_j \quad (j = 1, \dots, M)$ (3).

and $N(y_t; \theta_j, \Sigma_j)$ denotes the multivariate normal density function with mean θ_j and covariance matrix Σ_j. The parameter vector Θ now consists of the elements of the θ_j, the A_j, and the Ψ_j, along with the mixing proportions π_j $(j = 1, \dots, M-1)$, on putting $\pi_M = 1 - \sum_{j=1}^{M-1} \pi_j$. In the above formulation, the factors f_j are specific to the components, but the EM algorithm can be implemented with no extra difficulty without this specification to give the same results.

To estimate MFA parameters for the observed data y_t it's possible to use the multicycle Alternating Expectation Conditional Maximization (AECM) algorithm. The latter is an extended version of the EM algorithm and was proposed by Meng and Van Dyk (1997).

An implementation of the AECM algorithm to the MFA model requires decomposing the vector of parameters Θ in two parts as $(\Theta_1', \Theta_2')'$, where Θ_1 contains the mixing proportions π_j $(j = 1, \dots, M-1)$ and the elements of the component means θ_j $(j = 1, \dots, M)$. The subvector Θ_2 contains the elements of the A_j and the Ψ_j $(j = 1, \dots, M)$.

We let $\Theta^{(i)} = \left(\Theta_1^{(i)\prime}, \Theta_2^{(i)\prime}\right)$ be the value of Θ after the i-th iteration of the AECM algorithm. For this application of the AECM algorithm, one iteration consists of two cycles, and there is one E-step and two CM-steps for each cycle. The two CM-steps correspond to the partition of Θ into the two subvectors Θ_1 and Θ_2.

For the first cycle of the AECM algorithm, we specify the missing data to be just the component-indicator vectors, $\omega_1, \omega_2, \dots, \omega_T$, where $\omega_{jt} = (\omega_t)_j$ is one or zero, according to whether y_t arose or did not arise from the j-th component $(j = 1, \dots, M; t = 1, \dots, T)$. In this conceptualization of the mixture model, it is valid to assume that the observation y_t has arisen from one of the M components. For the second cycle for the updating of Θ_2, we specify the missing data to be the common latent factors $f_{j1}, f_{j2}, \dots, f_{jT}$, as well as the component-indicator labels ω_{jt}.

E-step

In order to carry out the E-step, we need to be able to compute the conditional expectation of the sufficient statistics. To carry out this step, we need to be able to calculate the conditional expectations,

$$E\{\omega_{jt} y_t f_{jt}' | y_t\}$$

and

$$E\{\omega_{jt} f_{jt} f_{jt}' | y_t\}$$

It follows from Rubin and Thayer (1982) that the conditional distribution of f_{jt} given y_t and $\omega_{jt} = 1$ is given by

$$(f_t | y_t, \omega_{jt} = 1) \sim N(\gamma_j'(y_t - \theta_j), \Omega_j) \quad (4)$$

for $j = 1, \dots, M$; $t = 1, \dots, T$, where

$$\gamma_j = (A_j A_j' + \Psi_j)^{-1} A_j$$

and where

$$\Omega_j = I_k - \gamma_j' A_j$$

Using (4),

$$E\{\omega_{jt} y_t f_{jt}' | y_t\} = \tau_j(y_t; \Theta) y_t (y_t - \theta_j)' \gamma_j$$

and

$$E\{\omega_{jt} f_{jt} f_{jt}' | y_t\} = \tau_j(y_t; \Theta) \gamma_j' \left\{(y_t - \theta_j)(y_t - \theta_j)' \gamma_j + \Omega_j\right\}$$

where $\tau_j(y_t; \Theta)$ is the j-th component-posterior probability of y_t defined by

$$\tau_j(y_t; \Theta) = \frac{\pi_j N(y_t; \theta_j, \Sigma_j)}{\sum_{l=1}^{M} \pi_l N(y_t; \theta_l, \Sigma_l)}$$

CM-steps

The first conditional CM-step leads to $\pi_j^{(i)}$ and $\theta_j^{(i)}$ being updated to

$$\pi_j^{(i+1)} = \frac{1}{T} \sum_{t=1}^{T} \tau_j(y_t; \Theta^{(i)})$$

and

$$\theta_j^{(i+1)} = \frac{\sum_{t=1}^{T} \tau_j(y_t; \Theta^{(i)}) y_t}{\sum_{t=1}^{T} \tau_j(y_t; \Theta^{(i)})}$$

for $j = 1, \dots, M$.

For the second cycle for the updating of Θ_2, we specify the missing data to be the latent factors $f_{j1}, f_{j2} \dots, f_{jT}$, as well as the component-indicator vectors, $\omega_1, \omega_2, \dots, \omega_T$. On setting $\Theta^{(i+1/2)}$ equal to $\left(\Theta_1^{(i+1)\prime}, \Theta_2^{(i)\prime}\right)'$, an E-step is performed to calculate $Q(\Theta, \Theta^{(i+1/2)})$, which is the conditional expectation of the complete-data log-likelihood given the observed data, using $\Theta = \Theta^{(i+1/2)}$. The CM-step on this second cycle is implemented by the maximization of $Q(\Theta, \Theta^{(i+1/2)})$ over Θ with Θ_1 set equal to $\Theta_1^{(i+1)}$. This yields the updated estimates $A_j^{(i+1)}$ and $\Psi_j^{(i+1)}$. The former is given by

$$A_j^{(i+1)} = v_j^{(i+1/2)} \gamma_j^{(i)} \left(\gamma_j^{(i)\prime} v_j^{(i+1/2)} \gamma_j^{(i)} + \Omega_j^{(i)} \right)^{-1}$$

where

$$v_j^{(i+1/2)} = \frac{\sum_{t=1}^{T} \tau_j\left(y_t; \Theta^{(i+1/2)}\right)\left(y_t - \theta_j^{(i+1)}\right)\left(y_t - \theta_j^{(i+1)}\right)'}{\sum_{t=1}^{T} \tau_j\left(y_t; \Theta^{(i+1/2)}\right)}$$

$$\gamma_j^{(i)} = \left(A_j^{(i)} A_j^{(i)\prime} + \Psi_j^{(i)}\right)^{-1} A_j^{(i)}$$

and

$$\Omega_j^{(i)} = I_k - \gamma_j^{(i)\prime} A_j^{(i)}$$

for $j = 1, \ldots, M$. The updated estimate $\Psi_j^{(i+1)}$ is given by

$$\Psi_j^{(i+1)} = diag\left\{ v_j^{(i+1/2)} - A_j^{(i+1)} H_j^{(i+1/2)} A_j^{(i+1)\prime} \right\}$$

where

$$H_j^{(i+1/2)} = \frac{\sum_{t=1}^{T} \tau_j\left(y_t; \Theta^{(i+1/2)}\right) E_j^{(i+1/2)}[f_t f_t' | y_t]}{\sum_{t=1}^{T} \tau_j\left(y_t; \Theta^{(i+1/2)}\right)} = \gamma_j^{(i)\prime} v_j^{(i+1/2)} \gamma_j^{(i)} + \Omega_j^{(i)}$$

and $E_j^{(i+1/2)}$ denotes conditional expectation given membership of the *j*-th component, using $\Theta^{(i+1/2)}$ for Θ.

2- Monte Carlo for Value at Risk

The Monte Carlo simulation is recognized as the optimal quantitative methodology for measuring the Value at Risk. Alexander (2008) showed that Monte Carlo VaR approach is very flexible and several assumptions can be attributed to the multivariate distribution of risk factor returns. This simulating method is able to detect or/and specify the possible changes in the market risk factors through the employment of a statistical distribution.

Ben Rejeb et al (2012), made a comparative analysis between four risk measurement models: Historical simulation, Variance Covariance, Bootstrapping and Monte Carlo simulation, to assess the foreign exchange risk in the Tunisian exchange market. In their empirical analysis, they found that VaR estimates related to three currencies (USD, EUR, JPY) from Monte Carlo simulation approach and Bootstrapping method were very similar. The results showed also that at a confidence level of 95%, VaR change depending on the simulation methods.

The basic problem of this study, is determining VaR for a portfolio of exchange rates via Monte Carlo simulation. The latter aims at generating risk measures through a statistical model. Our contribution is that this simulating method uses MFA model to generate different scenarios for the risk factors and combine these scenarios to generate correlated and heterogeneous future returns. The return of the portfolio at present time t will be denoted by R_t^p. Let us assume that R_t^p depends on q risk factors, then the main steps for doing this VaR estimation are as follows:

Algorithm

1.Select parameters of VaR: holding Period as well as confidence level$(1-\alpha)$.

2.Simulate the evolution of the common latent risk factors f_t^s from time t to time $t+1$ by generating q-tuples of pseudo random numbers from $N(0, I_k)$.

3.Simulate the evolution of the mixed specific risk factors ε_t from time t to time $t+1$ by generating q-tuples of pseudo random numbers with appropriate joint distribution that describes the behavior of the specific risk factors.

(a)Generate ε_t^* from $N(0, I_q)$.

(b)Generate V from $U(0,1)$.

(c)Return

$$\varepsilon_t^s = \sum_{j=1}^{M} \left(\hat{\theta}_j + \hat{\Psi}_j^* \cdot \hat{\varepsilon}_t^*\right) \mathbb{I}_{\left\{\sum_{l=1}^{j-1} \hat{\pi}_l \le U < \sum_{l=1}^{j} \hat{\pi}_l\right\}}$$

where $\sum_{j=1}^{0} \hat{\pi}_j = 0$ and $\hat{\Psi}_j^*$ is a lower triangular matrix obtained from the Cholesky decomposition of $\hat{\Psi}_j$.

The number N of these q-tuples has to be large enough (typically$N = O(1000)$) to obtain sufficient statistics in step 5.

4.Calculate the N different portfolio returns at time $t+1$ using the values of the simulated q-tuples of the common latent and specific risk factors. Let us denote these returns by $R_{t+1,1}^p$, $R_{t+1,2}^p, \ldots, R_{t+1,N}^p$ where

$$R_{t+1,s}^p = \delta_1 y_{1t+1}^s + \delta_2 y_{2t+1}^s + \cdots + \delta_q y_{qt+1}^s$$

and $\delta_1, \delta_2, \ldots, \delta_q$ are the portfolio weights for the q assets, whose returns are simulated by the model

$$y_{t+1}^s = A_j f_t^s + \varepsilon_t^s$$

5.Ignore the fraction of the α worst returns $R_{t+1,s}^p$. The minimum of the remaining $R_{t+1,s}^p$ is then the VaR of the portfolio at time t. It will be denoted by VaR$(\alpha, t, t+1)$.

As soon as the time evolves from t to $t+1$, the real return of the (unchanged) portfolio changes from R_t^p. to R_{t+1}^p. With this data at hand, one can backtest VaR$(\alpha, t, t+1)$ by comparing it with R_{t+1}^p.

3- Evaluation models: Backtesting

These methods provided solutions for the comparison between the different approaches (Nieppola, 2009). These methods of Backtesting are analysed and estimated with special attention. They compared the Out-of-Sample VaR estimates with Christoffersen tests (Kupiec Test, Independence test and Joint test). Then, a principal function is reported on a binary loss function that treats any loss larger than the VaR estimate as an ‘exception’. In this case, it’s possible to see whether failure rates are in accordance with selected the confidence level (Mirjana Miletic and Sinisa Miletic, 2015) through the Unconditional Coverage test. In addition, an accurate VaR model shows the number of exceptions that are independent all the time: Independent test (Evers and Rohde, 2014). Finally, we examine both features of Conditionality and movements in data all the time through the joint test named Conditional Coverage test (Jorion, 2007).

a. Kupiec test or PoF test

Kupiec test also named the Proportion of Failure (PoF), it examines the unconditional coverage feature (Kupiec, 1995). In this test, the main parameters are n (number of exceptions) and T (total number of observations) to quantify $\hat{\alpha}$ (observed proportion of failures: $\hat{\alpha} = n/T$) as well as the confidence level α (expected proportion of failure). The null hypothesis is that the observed probability of exception occurring is equal to the expected.

$$H_0: \alpha = \hat{\alpha}$$

The PoF test statistic is calculated by the likelihood Ratio (LR_{PoF}), the latter function can be written as:

$$LR_{PoF} = -2\ln\left[\frac{\alpha^n(1-\alpha)^{T-n}}{\hat{\alpha}^n(1-\hat{\alpha})^{T-n}}\right] \sim \chi^2(1)$$

b. Independence test

It takes into account the independence of exceptions. The independence test statistic is evaluated by the likelihood Ratio (LR_{ind})

$$LR_{ind} = -2\ln\left[\frac{(1-\pi)^{n_{00}+n_{10}}\pi^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}}\pi_0^{n_{01}}(1-\pi_1)^{n_{10}}\pi_1^{n_{11}}}\right] \sim \chi^2(1)$$

The null hypothesis is that the probability of exception occurring is independent on the information whether the exception has occurred also previous day:

$$H_0: \pi_0 = \pi_1$$

where π_i is the probability of having a failure conditional on state i and on previous day and n_{ij} $(i = 0,1\,; j = 0,1)$ is the number of days in which j is achieved in one day, however i was at the previous day.

$$\pi_0 = \frac{n_{01}}{n_{00}+n_{01}}$$

$$\pi_1 = \frac{n_{11}}{n_{10}+n_{11}}$$

$$\pi = \frac{(n_{01}+n_{11})}{(n_{00}+n_{01})+(n_{10}+n_{11})}$$

c. Conditional Coverage test (Joint Test)

According to this statistical test, an accurate VaR model includes both unconditional coverage and independence between exceptions (Christoffersen, 1998).

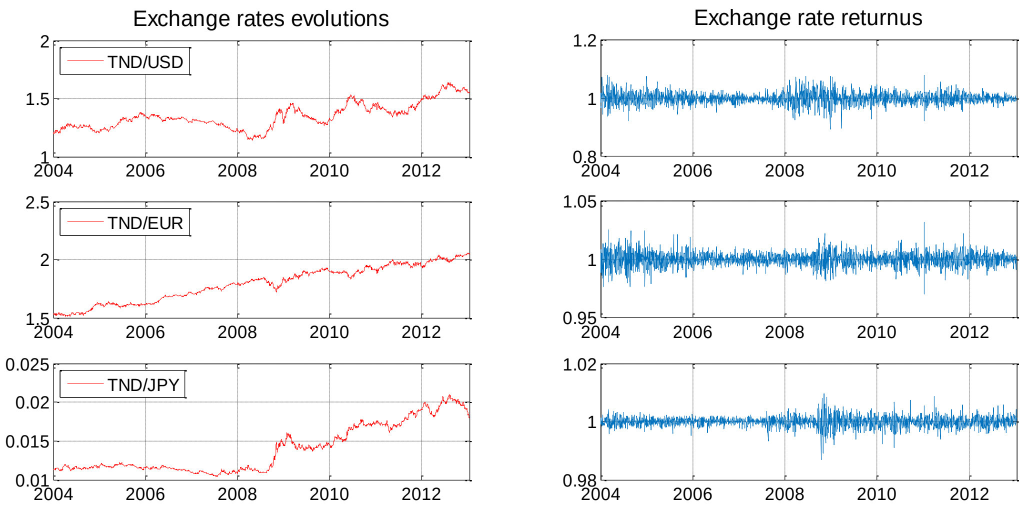
$$LR_{cc} = LR_{PoF} + LR_{ind} \sim \chi^2(2)$$

4. THE DATASET: TUNISIAN FOREIGN EXCHANGE RATES

4.1. Data Description

The models presented in the empirical literature are applied to a portfolio composed by the most representative currencies, in the same approach for the Tunisian context, we have opted for the most three foreign rates treated into the Tunisian FX, namely: TND/USD, TND/EUR and TND/JPY. The data set contains 2264 daily exchange rates from 05-01-2004 to 31-12-2012. Our sample consists of a long data that includes periods of sensible fluctuation and thus enables us to examine how the MFA approach perform during such periods. We have opted for daily sampling frequencies. The exchange rate series were extracted from an historical exchange database provided by the Tunisian central bank and FX database. In order, to evaluate VaR models, exchange rates are transformed into log-returns.

Figure 1. Real daily observed exchange rates and their returns from 05-01-2004 to 31-12-2012



As can be seen, figure 1, illustrates the movements of log-returns for exchange rates. In this case, it becomes very clear that the Tunisian currency returns changes as its volatility changes. Therefore, higher periods of volatility are followed by higher periods of low returns and vice versa, also we detect the presence of a co-movement between the different FX rates. This relationship is relatively apparent during the observation period and it is the central fact of our investigation into the Tunisian FX market. We see that the behaviour of log-returns for Tunisian currencies is highly volatile between 2008 and at the beginning of 2009. As mentioned above, this period is a considered as a transitional step to a managed floating exchange rate regime. Such period of transition is characterized by a series of significant changes that lead to significant volatility and repetitive shocks.

4.2. Descriptive Statistics

In this part, we describe statistical features of log-returns related to exchange rates using descriptive statistics, which are presented in table 1:

Table 1. Descriptive statistics of daily log-returns of Tunisian exchange rates

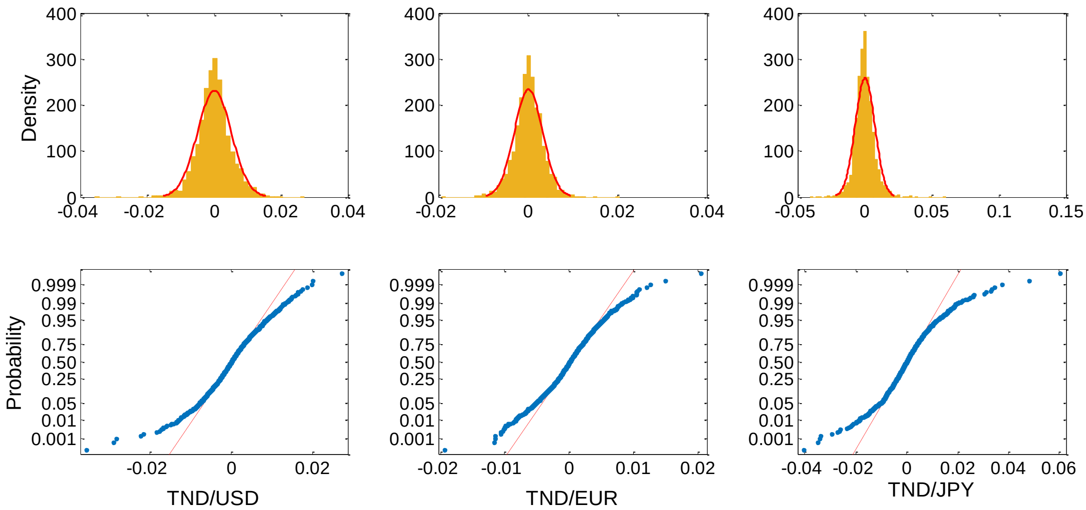
Statistics	TND/USD	TND/EUR	TND/JPY
Mean	0.0001098	0.0001314	0.00020715
Standard Deviation	0.0046554	0.0023225	0.00900527
Minimum Returns	-0.0327758	-0.0116711	-0.04463322
Maximum Returns	0.02603510	0.01093957	0.06323042
Kurtosis	6.11176	4.87366	5.72289
Skewness	0.12061	-0.12260	0.25772
Number of observations	2264	2264	2264
Jarque-Béra Statistic	918.9234	336.8402	724.4615

The table demonstrates that log-return series of exchange rates have positive mean daily returns. Then, the returns of TND/USD and TND/JPY were positively skewed, but they are negative for the TND/EUR. The null hypothesis for skewness coefficients that conform to a normal distribution's value of zero has been rejected at the 5% significance level; negative skewness indicates that the distribution has a long left tail, which indicates a high probability of observing large negative values. In addition, the returns for all currencies also exhibit excess kurtosis, particularly for TND/USD and TND/JPY.

The null hypothesis for kurtosis coefficients that conform to the normal value of three is rejected for all exchange rates; which is one of the Tunisian FX market features that exhibit important kurtosis. According to Jarque-Béra normality test, the null hypothesis of normality is rejected (for 95% significance level, critical value is 5.991).

In Figure 2 we depict the histograms of individual time series. For each histogram, we also superimpose the normal density function with the same mean and variance. Also plotted in Figure 2 are the normal probability plots for the three returns. The purpose of a normal probability plot is to graphically assess whether the data could come from a normal distribution. If the data are normal the plot will be linear. Other distribution types will introduce curvature in the plot. It is clear from this figure that the returns within the given holding periods are not normally distributed. Especially the tails of the return distributions are heavier than those of the normal distribution, which is highlighted explicitly in the normal probability plots: the left tail (red points) is above (larger) the red line, and the right tail (blue points) is below (negatively larger) the red line.

Figure 2. Histograms (the top panels) and the Normal probability plots (the bottom panels) of the daily, log return series from 05-01-2004 to 31-12-2012.



The normal density with the same mean and variance are superimposed on the histogram plots

4.3. Tests of Stationarity

We apply statistical tests to confirm the stationarity property for our log return series, namely: KPSS (Kwiatkowski–Phillips–Schmidt–Shin, 1992) tests: aims at testing the null hypothesis that observable time series is stationary, and PP (Phillips Perron, 1988) test: used to test the null hypothesis of unit root (Non stationary time series).

Table 2. Tests for stationarity

Currency	KPSS	PP
TND/USD	0.0542	-45.4211
TND/EUR	0.0154	-48.0820
TND/JPY	0.0532	-59.0684
The critical values at 5% level are equal to 0.1460 for the KPSS test and -1.9416 for PP test.		

Table 2, illustrates the results of KPSS test for the logarithmic returns for which the null hypothesis could not be rejected. However, when comparing PP statistics as specified in the same table with critical value of 5% level, the null hypothesis of unit root could be rejected. By these two statistical tests, we conclude that log-returns of Tunisian currencies are stationary. All these results highlight the usefulness of MFA, which take into account the stationarity, and the heterogeneous and asymmetric return distributions of the Tunisian currencies.

4.4. Correlation Analysis

In order to explain the interdependence between movements of the FX rates, we use correlation coefficient. The results from 2004 to 2012 can be seen in the tables below:

Table 3. Correlation matrix for daily foreign exchange rates log-returns

	TND/USD	TND/EUR	TND/JPY
TND/USD	1.0000	-0.6126	0.2973
TND/EUR	-0.6126	1.0000	-0.2814
TND/JPY	0.2973	-0.2814	1.0000

TND/USD and TND/EUR have a strong negative correlation(-0.6126), this coefficient demonstrates that these two FX rates are not behaving similarly. In other words, this can be expressed as follows: TND appreciates versus USD as well as depreciates versus EUR and vice versa. Therefore, the gain obtained from one currency (TND/USD) will recover the loss of the other currency (TND/EUR). Since the correlation coefficient of TND/JPY and TND/EUR (-0.2814), we understand the opposite movements of the Euro and JPY currencies. Finally, TND/USD and TND/JPY have a quite positive correlation coefficient(0.2973), one FX rates increases simultaneously with the other.

5. EMPIRICAL RESULTS: IDENTIFICATION OF THE BEST VAR MODEL

The objective of the latest part is to identify the most appropriate approach that can be adopted to forecast the VaR for the portfolio of Tunisian FX rates. However, we need first to examine the performance of VaR models by applying the method of rolling sample. To estimate the VaR, we divide dataset of Tunisian FX rates returns into two Parts: In Sample, in which the estimated period is from 02/01/2004 until 12/01/2005, (264 observations) and the Out-of-Sample - also called the test period- that begins on 13/01/2005 and ends on 31/12/2012 (2000 observations). To establish the VaR number of the public Tunisian external debt portfolio for a given confidence level ($1-\alpha$), we used the following portfolio weights: $\delta_1 = 14\%$ (for the TND/USD), $\delta_2 = 68\%$ (for the TND/EUR) and $\delta_3 = 18\%$ (for the TND/JPY).

The main difficulty in using Monte Carlo simulation for VaR inference is the amount of time that takes to compute correct estimates, especially, when the portfolio consists of many risk factors and/or when the confidence level is high. The problem with applying the MFA to VaR, is that many simulation paths are wasted in the sense that they are concentrated around current portfolio value, rather than being concentrated around the wanted VaR value. Therefore, with a rolling sample, we estimated parameters of VaR models (GMM, FA and MFA) using 1000 simulations for different risk levels (1%, 2% and 5%). The next step aims to check the stability and reliability of the results over time through the Backtesting procedure. In this case we apply Kupiec's PoF test, (Unconditional Coverage test), Independence test and Conditional Coverage test. The VaR numbers derived from the three approaches present a wide range of consequences. The results of the Backtesting are presented in the tables 4, 5 and 6.

Table 4. Model Evaluation for $\text{VaR}_\alpha = 1\%$

VaR Model	Failure rate	First Violation	LR_{TUFF}	LR_{PoF}	LR_{IND}	LR_{CC}
MFA	1.25	104	0.0016	1.1698	0.0000	1.1698
GMM	1.25	104	0.0016	1.1698	0.0000	1.1698
FA	2.50	104	0.0016	**32.08**	0.0566	**32.143**
LR_{critic} , $\alpha^* = 95\%$			3.841	3.841	3.841	5.991

Table 5. Model Evaluation for $\text{VaR}_\alpha = 2\%$

VaR model	Failure rate	First violation	LR_{TUFF}	LR_{PoF}	LR_{IND}	LR_{CC}
MFA	2.10	104	0.7067	0.1004	0.0157	0.1161
GMM	2.15	104	0.7067	0.2242	0.0000	0.2242
FA	3.60	104	0.7067	**21.167**	0.0656	**21.232**
LR_{critic} , $\alpha^* = 95\%$			3.841	3.841	3.841	5.991

Table 6. Model Evaluation for $VaR_{\alpha} = 5\%$

VaR model	Failure rate	First violation	LR_{TUFF}	LR_{PoF}	LR_{IND}	LR_{CC}
MFA	5.30	11	0.3153	0.3720	0.5737	0.9457
GMM	5.25	11	0.3153	0.2591	1.5067	1.7658
FA	7	53	1.4044	**15.060**	0.1629	**15.223**
LR_{critic} , $\alpha^* = 95\%$			3.841	3.841	3.841	5.991

Moreover, we can see in figures 3, 4 and 5 a faster volatility in 99%, 98% and 95% VaR estimates for the two mixtures models (MFA and GMM) than FA model. In addition, FA method gives VaR estimates lower than MFA and GMM approaches.

Figure 3. Out of Sample estimated 0.01, 0.02, 0.05 VaR and return portfolio for MFA model

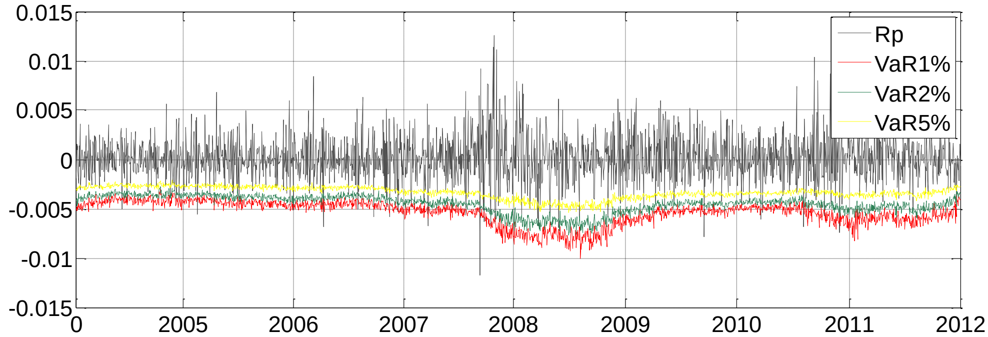


Figure 4. Out of Sample estimated 0.01, 0.02, 0.05 VaR and return portfolio for GMM model

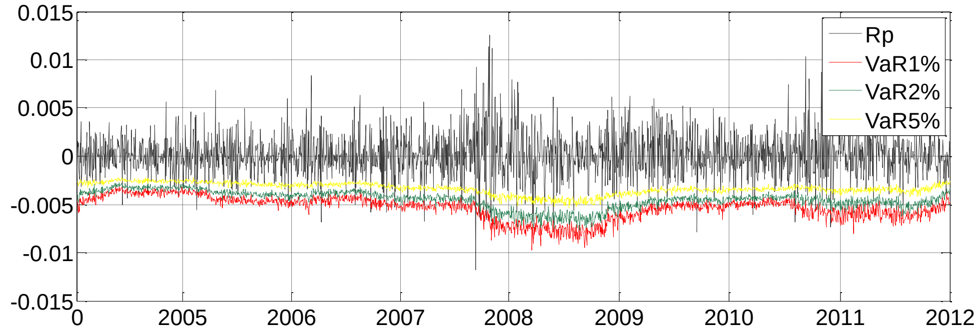
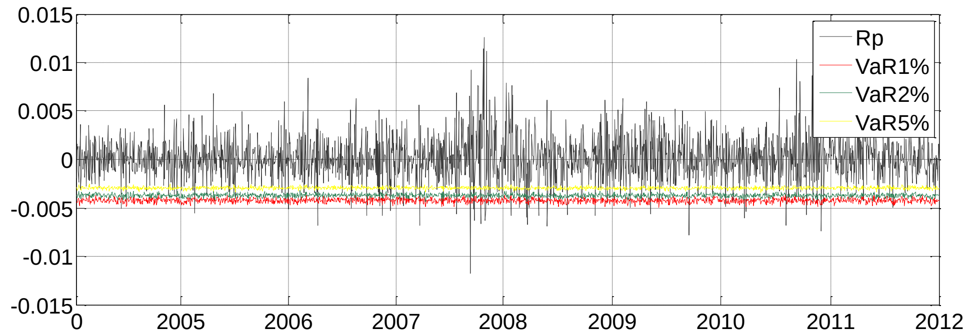


Figure 5. Out of Sample estimated 0.01, 0.02, 0.05 VaR and return portfolio for FA model



Evidently, VaR through MFA and GMM using Monte Carlo Simulation shows improvement in management of exchange risk exposure compared to the FA model over time. From year 2005 to 2012, as expected the FA method has poor results. Indeed, this approach underestimated risk, it presented poor results for the different risk levels 1%, 2% and 5% using the Christoffersen test (the test for conditional coverage was respectively 32.14, 21.23 and 15.22).

The VaR through MFA has suffered due to the fluctuation of the exchange rates of three currencies. Consequently, the results conclude in favour to reject such models credibility regarding its poor significant level. It is noticeable from the FX series that the VaR are affected by a significant fluctuation of the volatility (see figure 5). Thus, the main disadvantage of this method is its lack of reactivity, which suggest in part, the use of additional modelling structures that incorporates the interaction between the three risk factors (TND/USD, TND/EUR, TND/JPY). Such dependence can be represented by more adequate approaches that can estimate the Value at Risk in spite of the assumption of heterogeneity of the risk factors. We know that FA method explains the correlation between the variables observed from a minimum number of factors but it does not take into account the heterogeneity in the pattern of change across the different currencies. As noted before, MFA and GMM using Monte Carlo simulation are more suitable to construct the joint multivariate distribution of losses and are more flexible and realistic in terms of allowing a wide range of dependence structure. The MFA purposes to model the distribution of FX rates from a mixture of probabilistic factor analyzers, but The GMM aims to specify the distribution of the random variables using a mixture of Gaussian densities. The latter has the sum of several Gaussian densities, each one has its own parameters of μ & Σ.

This improvement in management of FX exchange risk was possibly mainly due to the changing of the FX regimes over time. Evidently, we have validated, for the three levels that MFA and GMM methods adjusted rapidly compared to FA (see figures 3 and 4), in order to better estimate the fluctuation of the volatility for the Out-of-Sample period and to detect sufficiently the fluctuation of the VaR. The first observation of tables 4, 5 and 6, and figures 3 and 4, shows that the VaR violations fluctuated with the volatility of the respective period. As mentioned earlier these models are able to capture adequately some particular characteristics of the portfolio series such as changes in volatility, heterogeneity and dependency.

It is noticed from the results that the detection of the first violation is improved. In fact, we saw that for the 5% risk level, both MFA and GMM models capture better losses on the 11-th day compared to the FA model 53-th day. However, for the level of 1% as well as 2% it is the 104-th day. For a 1% coverage rate, the results are similar for the MFA and GMM methods, more exactly the LRCC test gives the same significant test 1.16 and the identical Failure rate, which is estimated to 1.25 %. More precisely, the two VaR sequences (obtained from the MFA model and GMM method) are too similar for the backtests to discriminate between them.

From tables 5 and 6, the tests conclude to the validity of both risk measures 2% and 5% for the MFA model as well as for the GMM method. However, the results vary severely from one specification to another; more precisely, it is possible to give affirmative conclusions for the portfolio that MFA gives the best results in terms of Backtesting. Indeed, the unconditional and conditional coverage tests give better results than the GMM model.

Table 5 shows that MFA model perform better for the FX portfolio at the level 2% as the failure rate has been 2.10%. The MFA model appears to be remarkably accurate in that case compared to the GMM approach. Indeed, this approach provides accurate VaR forecasts for this portfolio at the confidence levels of 98%. Moreover, it has approved the two tests for “unconditional coverage” and for “independence” respectively (0.1004) and (0.0157), implying a significant Christoffersen Test (0.1161) which is inferior to χ^2 (5.99).

Table 6 confirms that results were linked to the previous findings in the risk level 2%. In fact, we found that MFA model performs better the forecast for the FX portfolio for the risk level 5% than the GMM model, because we have more significant LRCC (0.9457) compared to other method with LRCC (1.7658).

Finally, we can compare the Backtesting results using the following quantity:

$$S \equiv \sum_{i=1}^{3} \left(\frac{\hat{\alpha}_i - \alpha_i}{\alpha_i} \right)^2$$

where $(\hat{\alpha}_i)$ are the failure rates for each method. In this case, the best model is the one who have the smallest value S. Hence, using the quantity S we can classify our VaR models. It seems from table 7 that MFA is the most accurate and consistent approach to forecast correctly the VaR for currency risk, which means according to table 7 that it’s observed proportion of failure $\hat{\alpha}$ is very close to expected proportion of failure α (1%, 2% and 5%). GMM method is the second one, and finally we found that FA approach has the biggest S.

Table 7. Failure rate for α= 0.01, 0.02 and 0.05

α	MFA	GMM	FA
0.01	1.25	1.25	2.50
0.02	2.10	2.15	3.60
0.05	5.30	5.25	7
S	**0.0686**	0.070625	3.05

As a consequence, the comparison between different models with each specification shows that, according to the different measures used for the performance of failure rates forecast and then the Backtesting of the Value at Risk, the MFA and GMM approaches provide the best Out-of-Sample estimation for the risk levels 1%, 2% and 5% for the Tunisian FX market. MFA is ranked in the first position as the results demonstrate the presence of the lower exceptions and the more significant tests, for the different risk levels than the other competitor models. The GMM was ranked in the second position for the level 2% and 5%.

CONCLUSION

Modeling of financial time series is clearly difficult but not less important part of financial risk management. The difficulties of modeling are caused by the specific characteristics of financial time series, such as heterogeneity, fat tails, volatility clustering and dependence, which cannot be easily modeled. In this paper a new methodology based on the mixture of probabilistic factor analyzers model were proposed and backtested on the Tunisian public debt portfolio. Assuming risk levels of 1%, 2% and 5%, we have calculated the VaRs for this portfolio on the basis of 264 banking days, using the GMM, the FA model and our proposed MFM model. The computations and the corresponding Backtesting of the results have been performed on the basis of historical foreign exchange rates ranging over eight years. This means that our Backtesting statistics are based on approximately 2000 measurements.

More precisely, to forecast VaR though these approaches, analysis is conducted on a test period from 13/01/2005 to 31/12/2012. We proved that MFA and GMM models give adequate VaR estimates and were the most accurate in assessing the Tunisian currency risk. Our results showed that the

MFA and GMM (particularly MFA) are ranked in the first position to perform an analysis of the Value at Risk for the 1%, 2% and 5% risk levels. Such models perform better than other approaches as they give a significant capital allocation and a well estimation of exceptions. This finding is tested empirically using Backtesting techniques under different tests with three level of risk. We notice also, that the FA model gives statistically insignificant estimations, it lacking the property of "correct conditional coverage" capital, thus, the results concluded in favor to reject such models credibility considering its poor significant level.

To be able to decide whether or not one should prefer the mixture of factor analyzers model to the traditional models, a supplementary investigation of the speed of the proposed algorithm would be useful. As we considered three risk factors only, a natural extension of studies in this field would have to include four or more risk factors. Furthermore, our model can be generalized to one where one allows the common latent factors and the specific factors to be stochastic functions of time. By combining the mixture of factor analyzers models with hidden Markov chain models, we can derive a dynamical local model for segmentation and prediction of multivariate conditionally heteroscedastic financial time series (see, for instance, Saidane and Lavergne, 2009, 2011, 2013). The study of such models would provide a further step in the extension of hidden Markov models to mixed conditionally heteroscedastic latent factor models and allow for further flexibility in the market risk analysis and value-at-risk applications.

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